IID: Maximum Principles

Consider now an elliptic operator in non-divergence form

$$L = -a^{jk}(x)\partial_j\partial_k + b^k(x)\partial_k + c(x)$$

with coefficients $a^{jk}, b^k, c \in C(\Omega) \cap L^\infty(\Omega)$ which are continuous and bounded on a bounded, open, connected subset $\Omega \subset \mathbb{R}^n$.

Theorem (maximum principle): suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

- $c(x) \equiv 0$: if $Lu \leq 0$ (subsolution) in Ω, then $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.

  If u attains its maximum at a point inside Ω, $u \equiv$ constant.
- $c(x) \geq 0$: if $Lu \leq 0$ in Ω, then $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+$.

  If u attains a *non-negative* maximum inside Ω, $u \equiv$ constant.

Here $u^+(x) := \max(u(x), 0)$ denotes the positive part of u. Of course an analogous statement applies for supersolutions ($Lu \geq 0$) and minima, by replacing $u \mapsto -u$.

Proof: we will first establish the *weak* maximum principle statements, starting with the case $c(x) \equiv 0$.

Supposing first that we have a *strict* subsolution, $Lu < 0$, the statement follows by elementary calculus: if u has a local maximum at $x_0 \in \Omega$, then $\nabla u(x_0) = 0$ and $D^2u(x_0) \leq 0$. Using the ellipticity $A(x) = \{a^{jk}(x)\}_{jk=1}^n \geq \theta > 0$, some simple linear algebra shows that also $a^{jk}(x_0)u_{x_jx_k}(x_0) \leq 0$, so $Lu(x_0) \geq 0$, a contradiction.

Now suppose only $Lu \leq 0$ in Ω. Given $\varepsilon > 0$, set $u_\varepsilon(x) := u(x) + \varepsilon e^{\lambda x_1}$ for some $\lambda > 0$ to be chosen. Compute

$$Lu_\varepsilon = Lu + \varepsilon e^{\lambda x_1}\left(-\lambda^2 a^{11}(x) + \lambda b^1(x)\right) \leq \varepsilon e^{\lambda \min(x_1)}\left(-\lambda^2\theta + \lambda\|b^1\|_{L^\infty}\right) < 0$$

in Ω for λ large enough. So by the above,

$$\max_{\bar{\Omega}} u \leq \max_{\bar{\Omega}} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon,$$

and then letting $\varepsilon \searrow 0$, $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u$.

Now suppose $c(x) \geq 0$. If $u(x) \leq 0$, we are done. Otherwise, consider the shifted operator $K := L - c$ on the non-empty open set $V := \{x \in \Omega \mid u(x) > 0\}$, on which $Ku = Lu - cu \leq 0$. So by the above, $\max_V u \leq \max_{\partial V} u$. If $x \in \partial V$ then either $x \in \partial\Omega$ or else $u(x) = 0$, and so

$$\max_{\Omega} u = \max_{V} u = \max_{\partial\Omega} u^+.$$

To prove the strong maximum principles we need:

Hopf boundary lemma: suppose $u \in C^2(B) \cap C^1(\bar{B})$ satisfies $Lu \leq 0$ in an open ball $B \subset \Omega$, and suppose there is $x_0 \in \partial B$ such that $u(x) < u(x_0)$ for all $x \in B$. Finally, suppose either $c \equiv 0$, or $c \geq 0$ and $u(x_0) \geq 0$. Then $\frac{\partial u}{\partial \nu}(x_0) > 0$.

Here $\frac{\partial}{\partial \nu}$ denotes the derivative in the direction of the outward normal to ∂B. Note that $\frac{\partial u}{\partial \nu}(x_0) \geq 0$ is obvious – the content is in the strict inequality. We'll postpone its proof, and continue with the proof of the strong maximum principle.

Set $M := \max_{\bar{\Omega}} u$. Suppose $u \not\equiv M$, and consider the non-empty open set $V := \{x \in \Omega \mid u(x) < M\}$. Note the boundary of V is contained in the union of $\partial\Omega$ and the closed set $C := \{x \in \Omega \mid u(x) = M\}$. We assume C is non-empty and derive a contradiction. Choose $y \in V$ so that $dist(y, C) < dist(y, \partial\Omega)$, and let B be the largest open ball centred at y with $B \subset V$. Then there is $x_0 \in \partial B \cap C$. We apply the Hopf boundary lemma on B to conclude $\frac{\partial u}{\partial \nu}(x_0) > 0$, which contradicts the fact that u has a maximum at x_0 and so $\nabla u(x_0) = 0$. $\square$.

Proof of the Hopf lemma: fist note that we may assume $u(x_0) \geq 0$ and $c \geq c$, since if $c \equiv 0$ we may replace $u(x)$ by $u(x) + M$ for any M, noting that still $L(u + M) = Lu \leq 0$. Assume B is centred at 0 and of radius R. Define the comparison function

$$v(x) := e^{-\lambda|x|^2} - e^{-\lambda R^2}$$

on B, for λ to be chosen (large), and compute

$$\begin{aligned} Lv &= e^{-\lambda|x|^2}\left(-4\lambda^2 a^{jk} x_j x_k + 2\lambda a^{kk} - 2\lambda b^k x_k + c\right) - ce^{-\lambda R^2} \\ &\leq e^{-\lambda|x|^2}\left(-4\theta\lambda^2|x|^2 + 2\lambda\|a\|_{L^\infty} + 2\lambda\|b\|_{L^\infty}|x| + \|c\|_{L^\infty}\right) \end{aligned}$$

and so on the annulus $A := \{x \mid R/2 < |x| < R\}$,

$$Lv \leq e^{-\lambda|x|^2}\left(-\theta\lambda^2 R^2 + 2\lambda\|a\|_{L^\infty} + 2\lambda\|b\|_{L^\infty} R + \|c\|_{L^\infty}\right) < 0$$

if we choose λ large enough. Now for $\varepsilon > 0$, apply the weak maximum principle to the function $u(x) + \varepsilon v(x) - u(x_0)$ on A, noting that

$$L\left(u + \varepsilon v - u(x_0)\right) = Lu + \varepsilon Lv - cu(x_0) \leq 0,$$

$$\max_{R/2 \leq |x| \leq R}\left(u(x) + \varepsilon v(x) - u(x_0)\right) \leq \max_{\{|x|=R/2\}\cup\{|x|=R\}}\left(u(x) + \varepsilon v(x) - u(x_0)\right)^+.$$

Note that on $\{|x| = R\}$, $v(x) = 0$, so $u + \varepsilon v - u(x_0) \leq 0$, while on $\{|x| = R/2\}$, $v(x) = e^{-\lambda R^2/4} - e^{-\lambda R^2}$ and $\max u < u(x_0)$ (by compactness of the sphere), so for ε sufficiently small, $u + \varepsilon v - u(x_0) < 0$. Thus $u(x) + \varepsilon v(x) - u(x_0) \leq 0$ on A and vanishes at x_0, so

$$0 \leq \frac{\partial}{\partial \nu}\left(u + \varepsilon v - u(x_0)\right)(x_0) = \frac{\partial u}{\partial \nu}(x_0) + \varepsilon\frac{\partial v}{\partial \nu}(x_0).$$

Finally, $\frac{\partial v}{\partial \nu}(x_0) = -2\lambda R e^{-\lambda R^2} < 0$, so $\frac{\partial u}{\partial \nu}(x_0) > 0$. $\square$