IID: Maximum Principles

Consider now an elliptic operator in non-divergence form

\[ L = -a^{jk}(x)\partial_j\partial_k + b^k(x)\partial_k + c(x) \]

with coefficients \( a^{jk}, b^k, c \in C(\Omega) \cap L^\infty(\Omega) \) which are continuous and bounded on a bounded, open, connected subset \( \Omega \subset \mathbb{R}^n \).

**Theorem (maximum principle):** suppose \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \).

- \( c(x) \equiv 0 \): if \( Lu \leq 0 \) (subsolution) in \( \Omega \), then \( \max_{\Omega} u = \max_{\partial\Omega} u \).
  
  If \( u \) attains its maximum at a point inside \( \Omega \), \( u \equiv \text{constant} \).

- \( c(x) \geq 0 \): if \( Lu \leq 0 \) in \( \Omega \), then \( \max_{\Omega} u \leq \max_{\partial\Omega} u^+ \).
  
  If \( u \) attains a non-negative maximum inside \( \Omega \), \( u \equiv \text{constant} \).

Here \( u^+(x) := \max(u(x), 0) \) denotes the positive part of \( u \). Of course an analogous statement applies for supersolutions \( (Lu \geq 0) \) and minima, by replacing \( u \mapsto -u \).

**Proof:** we will first establish the weak maximum principle statements, starting with the case \( c(x) \equiv 0 \).

Supposing first that we have a strict subsolution, \( Lu < 0 \), the statement follows by elementary calculus: if \( u \) has a local maximum at \( x_0 \in \Omega \), then \( ru_u(x_0) = 0 \) and \( D^2u(x_0) \leq 0 \). Using the ellipticity \( A(x) = \{a^{jk}(x)\}_{jk=1}^n \geq \theta > 0 \), some simple linear algebra shows that also \( a^{jk}(x_0)u_{x_jx_k}(x_0) \leq 0 \), so \( Lu(x_0) \geq 0 \), a contradiction.

Now suppose only \( Lu \leq 0 \) in \( \Omega \). Given \( \varepsilon > 0 \), set \( u_\varepsilon(x) := u(x) + \varepsilon e^{\lambda x_1} \) for some \( \lambda > 0 \) to be chosen. Compute

\[ Lu_\varepsilon = Lu + \varepsilon e^{\lambda x_1} (-\lambda^2 a^{11}(x) + \lambda b^1(x)) \leq \varepsilon e^{\lambda \min(x_1)} (-\lambda^2 \theta + \lambda \|b^1\|_{L^\infty}) < 0 \]

in \( \Omega \) for \( \lambda \) large enough. So by the above,

\[ \max_{\Omega} u \leq \max_{\Omega} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon, \]

and then letting \( \varepsilon \downarrow 0 \), \( \max_{\Omega} u \leq \max_{\partial\Omega} u \).

Now suppose \( c(x) \geq 0 \). If \( u(x) \leq 0 \), we are done. Otherwise, consider the shifted operator \( K := L - c \) on the non-empty open set \( V := \{x \in \Omega \mid u(x) > 0\} \), on which \( Ku = Lu - cu \leq 0 \). So by the above, \( \max_V u \leq \max_{\partial V} u \). If \( x \in \partial V \) then either \( x \in \partial \Omega \) or else \( u(x) = 0 \), and so

\[ \max_{\Omega} u = \max_{V} u = \max_{\partial V} u^+. \]

To prove the strong maximum principles we need:
Hopf boundary lemma: suppose $u \in C^2(B) \cap C^1(\bar{B})$ satisfies $Lu \leq 0$ in an open ball $B \subset \Omega$, and suppose there is $x_0 \in \partial B$ such that $u(x) < u(x_0)$ for all $x \in B$. Finally, suppose either $c \equiv 0$, or $c \geq 0$ and $u(x_0) \geq 0$. Then $\frac{\partial u}{\partial v}(x_0) > 0$.

Here $\frac{\partial}{\partial v}$ denotes the derivative in the direction of the outward normal to $\partial B$. Note that $\frac{\partial u}{\partial v}(x_0) \geq 0$ is obvious – the content is in the strict inequality. We’ll postpone its proof, and continue with the proof of the strong maximum principle.

Set $M := \max_{\Omega} u$. Suppose $u \not\equiv M$, and consider the non-empty open set $V := \{x \in \Omega \mid u(x) < M\}$. Note the boundary of $V$ is contained in the union of $\partial \Omega$ and the closed set $C := \{x \in \Omega \mid u(x) = M\}$. We assume $C$ is non-empty and derive a contradiction. Choose $y \in V$ so that $\text{dist}(y, C) < \text{dist}(y, \partial \Omega)$, and let $B$ be the largest open ball centred at $y$ with $B \subset V$. Then there is $x_0 \in \partial B \cap C$. We apply the Hopf boundary lemma on $B$ to conclude $\frac{\partial u}{\partial v}(x_0) > 0$, which contradicts the fact that $u$ has a maximum at $x_0$ and so $\nabla u(x_0) = 0$. $\square$

Proof of the Hopf lemma: first note that we may assume $u(x_0) \geq 0$ and $c \geq c$, since if $c \equiv 0$ we may replace $u(x)$ by $u(x) + M$ for any $M$, noting that still $L(u + M) = Lu \leq 0$. Assume $B$ is centred at 0 and of radius $R$. Define the comparison function

$$v(x) := e^{-\lambda|x|^2} - e^{-\lambda R^2}$$

on $B$, for $\lambda$ to be chosen (large), and compute

$$Lu = e^{-\lambda|x|^2} \left(-4\lambda^2a_{jk}x_jx_k + 2\lambda a^{kk}x_k + c\right) - ce^{-\lambda R^2}
\leq e^{-\lambda|x|^2} \left(-4\lambda^2|x|^2 + 2\lambda \|a\|_{L^\infty} + 2\lambda \|b\|_{L^\infty} |x| + \|c\|_{L^\infty}\right)$$

and so on the annulus $A := \{x \mid R/2 < |x| < R\},$

$$Lu \leq e^{-\lambda|x|^2} \left(-\theta^2 R^2 + 2\lambda \|a\|_{L^\infty} + 2\lambda \|b\|_{L^\infty} R + \|c\|_{L^\infty}\right) < 0$$

if we choose $\lambda$ large enough. Now for $\varepsilon > 0$, apply the weak maximum principle to the function $u(x) + \varepsilon v(x) - u(x_0)$ on $A$, noting that

$$L(u + \varepsilon v - u(x_0)) = Lu + \varepsilon Lv - cu(x_0) \leq 0,$$

$$\max_{R/2 \leq |x| \leq R} (u(x) + \varepsilon v(x) - u(x_0)) \leq \max_{\{|x| = R/2\} \cup \{|x| = R\}} (u(x) + \varepsilon v(x) - u(x_0))^+. $$

Note that on $\{|x| = R\}$, $v(x) = 0$, so $u + \varepsilon v - u(x_0) \leq 0$, while on $\{|x| = R/2\}$, $v(x) = e^{-\lambda R^2/4} - e^{-\lambda R^2}$ and max $u < u(x_0)$ (by compactness of the sphere), so for $\varepsilon$ sufficiently small, $u + \varepsilon v - u(x_0) < 0$. Thus $u(x) + \varepsilon v(x) - u(x_0) \leq 0$ on $A$ and vanishes at $x_0$, so

$$0 \leq \frac{\partial}{\partial v} (u + \varepsilon v - u(x_0))(x_0) = \frac{\partial u}{\partial v}(x_0) + \varepsilon \frac{\partial v}{\partial v}(x_0).$$

Finally, $\frac{\partial u}{\partial v}(x_0) = -2\lambda Re^{-\lambda R^2} < 0$, so $\frac{\partial u}{\partial v}(x_0) > 0$. $\square$