Boundary Regularity

Theorem (H^2 regularity up to the boundary): suppose $a^{jk} \in C^1(\bar{\Omega})$, b^j, $c \in L^\infty(\Omega)$. We assume also $\partial\Omega$ is C^2 (locally the graph of a C^2 function). Then there is $C > 0$ (depending on Ω and the coefficients) such that any weak solution $u \in H_0^1(\Omega)$ of the Dirichlet BVP $\left\{ \begin{array}{cc} Lu = f & \Omega \\ u = 0 & \partial\Omega \end{array} \right\}$ with $f \in L^2(\Omega)$, satisfies $u \in H^2(\Omega)$, and

$$\|u\|_{H^2(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right).$$

Remarks:

- the $\|u\|_{L^2}$ on the right side of the estimate is seen to be necessary by considering the case when there is non-uniqueness of solutions (the second Fredholm alternative), so any multiple of a non-trivial homogeneous solution can be added to u. It can be shown that in the unique case (first Fredholm alternative) that $\|u = L^{-1}f\|_{L^2(\Omega)} \lesssim \|f\|_{L^2}$, and then the estimate above reads $\|u = L^{-1}f\|_{H^2(\Omega)} \lesssim \|f\|_{L^2}$;
- one can also establish higher boundary regularity under more smoothness assumptions on the coefficients and Ω – see, eg, [Evans].

Main ideas of proof (see eg. [Evans] for details: first consider the special case of a half-ball

$$\Omega = B(0,1) \cap \{x_n > 0\},$$

and study the regularity up to the *flat part* of the boundary

$$\bar{B}_{\mathbb{R}^{n-1}}(0,1) \times \{x_n = 0\}.$$

So we may think of the solution u as vanishing in a neighbourhood of the *curved part* of the boundary (but not the flat part) – in practice, this is achieved using a cut-off function. Under this assumption, derivatives ∂_{x_l}, $l \neq n$ of u in the directions *not* normal to the flat boundary all vanish on $\partial\Omega$, so an integration by parts argument like the one used for interior regularity (made rigorous by replacing derivatives by finite differences and then taking a limit) yields

$$\sum_{(l,m)\neq(n,n)} \|u_{lm}\|^2_{L^2(\Omega)} \lesssim \|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}.$$

To deal with the second-derivative u_{nn} normal to the flat boundary, use the PDE (in non-divergence form)

$$a^{nn} u_{nn} = - \sum_{(j,k)\neq(n,n)} a^{jk} u_{jk} + (b^j - a^{jk}_k) u_j + cu - f.$$

Note that $a^{nn}(x) \geq \theta > 0$ by the ellipticity (use $\xi = \hat{e}_n$). Note also by the interior regularity, that this equation makes sense (at least almost everywhere). Then we may bound

$$|u_{nn}| \lesssim \sum_{(j,k)\neq(n,n)} |u_{jk}| + |\nabla u| + |u| + |f|$$

and so get the missing estimate

$$\|u_{nn}\|_{L^2} \lesssim \sum_{(j,k)\neq(n,n)} \|u_{jk}\|_{L^2} + \|\nabla u\|_{L^2} + \|u\|_{L^2} + \|f\|_{L^2}$$

from which follows

$$\|u\|_{H^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}.$$

To deal with an actual curved boundary piece which is given (by assumption) by the graph of a C^2 function, a change of variables is used to map this portion of the region into a half-ball. This change of variables preserves ellipticity, and the above half-ball estimate can be used, and the desired estimate recovered for that portion of Ω.

Finally, by compactness of $\partial\Omega$, it can be exhausted by finitely many such pieces on which this change-of-variables procedure may be used. This gives the desired global estimate. $\square$