IIC: Regularity

Having discussed the existence of weak solutions $u \in H_0^1(\Omega)$ of the Dirichlet boundary value problem

$$\begin{cases} Lu = f & \Omega \\ u = 0 & \partial\Omega \end{cases} \qquad (BVP)$$

where L is the uniformly elliptic, divergence-form operator

$$L = -\partial_j a^{jk}(x)\partial_k + b^j(x)\partial_j + c(x), \qquad \exists\theta > 0 \quad a^{jk}\xi_j\xi_k \geq \theta|\xi|^2 \ \forall x \in \Omega, \ \forall \xi \in \mathbb{R}^n.$$

The next question is how *smooth* (*regular*) are such solutions – for example, are they $C^2(\Omega)$ and therefore solutions in the classical sense?

The natural way to measure regularity here is via **Sobolev spaces** and **Sobolev norms**. It is convenient to use *multi-index notation* for partial derivatives:

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), 0 \leq \alpha_j \in \mathbb{Z}, \quad \partial^{(\alpha)} = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}, \quad |\alpha| = \alpha_1 + \alpha_2 + \cdots \alpha_n.$$

Recall:

for $0 \leq k \in \mathbb{Z}$, the **Sobolev norm**

$$\|f\|_{H^k(\Omega)} := \left(\sum_{|\alpha|\leq k} \|\partial^{(\alpha)} f\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$$

defines the **Sobolev space** (a Banach space)

$$H^k(\Omega) := \{f : \Omega \to \mathbb{R} \mid \|f\|_{H^k(\Omega)} < \infty\}.$$

The regularity question divides into two: regularity at interior points $x \in \Omega$; and regularity up to the boundary $\partial\Omega$.

<u>Interior Regularity</u>

A little notation:

- $L^\infty(\Omega)$ denotes the bounded functions on Ω (strictly speaking: measurable functions which are bounded except possibly on sets of zero measure) and its norm $\|f\|_{L^\infty(\Omega)} := \sup_{x\in\Omega} |f(x)|$ (strictly speaking, supremeum after possibly removing sets of measure zero);
- $V \subset\subset \Omega$ denotes V *compactly contained* in Ω: i.e. its closure $\bar{V} \subset \Omega$.

Theorem (H^2 interior regularity): suppose $a^{jk} \in C^1(\Omega)$, b^j, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$. Then $u \in H^2_{loc}(\Omega)$. Precisely: for any $V \subset\subset \Omega$, there is $C > 0$ (depending on V, Ω, coefficients) so that if $u \in H^1(\Omega)$ (weakly) solves $Lu = f$,

$$\|u\|_{H^2(V)} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right).$$

Remarks:

- note that boundary conditions play no role here;
- on first sight it might seem weird to have $\|u\|_{L^2}$ appear on the right-hand side, but since BCs play no role, imagine, eg., $L = -\Delta$ and $u(x) \equiv A$ is a constant;
- *gain of two derivatives* $f \in L^2 \implies u \in H^2$ is a typical *elliptic regularity* result;
- note that if $u \in H^2_{loc}$, then $Lu \in L^2$, and the original PDE holds as an equality of L^2 functions – in particular, is at least satisfied "almost everywhere".

Proof (sketch): first, as we did in the previous section, we get a mild regularity estimate just by using u itself as a test function in the weak form of the PDE, the ellipticity, Young's inequality, and Hölder's inequality:

$$\begin{aligned}
\|u\|_{L^2}\|f\|_{L^2} \geq (u, f) = B[u, u] &= \int_\Omega \left(a^{jk} u_j u_k + b^j u_j u + c u^2 \right) \, dx \\
&\geq \theta \|\nabla u\|^2_{L^2} - \int_\Omega \left(\frac{\theta}{2} |\nabla u|^2 + (\frac{1}{2\theta} \|b\|^2_{L^\infty} + \|c\|_{L^\infty}) u^2 \right) \, dx
\end{aligned}$$

and so arrive at

$$\|\nabla u\|^2_{L^2(\Omega)} \lesssim \|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)},$$

where we have used a notation very common in PDE analysis:

$A \lesssim B$ means $A \leq CB$ for some constant C (which is *fixed* for the estimate of interest).

As is completely typical in PDE analysis, the main ingredient is an *a priori estimate*: suppose u is in fact smooth (which we do not know yet), and compactly supported in Ω (which it is not!) to justify integrations by parts: multiply the PDE $Lu = f$ by the second derivative u_{ll} (for some $1 \leq l \leq n$), integrate by parts, and use the ellipticity again, and Young's inequality:

$$\begin{aligned}
0 &= \int_\Omega u_{ll} \left[-(a^{jk} u_k)_j + b^j u_j + cu - f \right] = \int_\Omega \left[u_{llj} a^{jk} u_k + b^j u_j u_{ll} + c u u_{ll} - f u_{ll} \right] \\
&= \int_\Omega \left[-a^{jk} u_{lj} u_{lk} - u_{lj} (a^{jk})_l u_k + b^j u_j u_{ll} + c u u_{ll} - f u_{ll} \right] \\
&\leq -\theta \int_\Omega |\nabla u_l|^2 + \int_\Omega \left[\frac{\theta}{2n} |D^2 u|^2 + \frac{n}{2\theta} \left((\|\nabla a\|^2_{L^\infty} + |b|^2_{L^\infty}) |\nabla u|^2 + \|c\|^2_{L^\infty} u^2 + f^2 \right) \right]
\end{aligned}$$

and so summing over l, and using our first estimate,

$$\|D^2u\|^2_{L^2(\Omega)} \lesssim \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} \lesssim \|u\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)}.$$

Now, here's how we can make this estimate rigorous. Fix $V \subset\subset \Omega$.

1. to justify the integration by parts without boundary terms, we will multiply throught by a smooth *cut-off function*:

$$\eta \in C_c^\infty(\Omega), \quad \eta \equiv 1 \text{ on } V$$

   (the existence of such an η is a fact from analysis, related to *Urysohn's lemma*);

2. to justify using u_{ll} (certainly not $\notin H_0^1(\Omega)$!) as a test function, we replace it with a finite-difference approximation:

$$u_{ll} \mapsto v := -D_l^{-h}\left(\eta^2 D_l^h\right) \in H_0^1(\Omega)\ , \qquad D_l^h u(x) := \frac{u(x+h\hat{e}_l) - u(x)}{h}$$

Inserting this v into the weak form of the PDE and "integrating by parts" as above, yields

$$\int_V |D^h \nabla u|^2 \lesssim \|u\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)},$$

and then taking $h \to 0$ yields

$$\int_V |D^2 u| \lesssim \|u\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)},$$

from which the desired estimate follows. □

By an iteration of this argument, we can obtain higher interior regularity, as long as the source term and coefficients are sufficiently smooth:

Theorem (higher interior regularity): let $m \in \mathbb{N}$. If $a^{jk}, b^j, c \in C^{m+1}(\Omega)$ and $f \in H^m(\Omega)$, then a (weak) solution of $Lu = f$ satisfies $u \in H^{m+2}_{loc}(\Omega)$ (with estimates as above).

We can relate Sobolev regularity back to classical regularity by way of the **Sobolev embedding theorem**: for $k \in \mathbb{N}$,

$$u \in H^m_{loc}(\Omega) \text{ for some } m > k + \frac{n}{2} \implies u \in C^k(\Omega).$$

So as a corollary of the higher regularity, we have:

- $a^{jk}, b^j, c \in C^{m+1}(\Omega)$, $f \in H^m(\Omega)$ for some $m > n/2 \implies u \in C^2(\Omega)$ is a classical solution of $Lu = f$;
- $a^{jk}, b^j, c \in C^\infty(\Omega)$, $f \in C^\infty(\Omega) \implies u \in C^\infty(\Omega)$ (this generalizes u harmonic $\implies u \in C^\infty$).