IIC: Regularity

Having discussed the existence of weak solutions $u \in H^1_0(\Omega)$ of the Dirichlet boundary value problem

$$\begin{cases} L u = f & \Omega \\ u = 0 & \partial \Omega \end{cases} \quad (BVP)$$

where $L$ is the uniformly elliptic, divergence-form operator

$$L = -\partial_j a^{jk}(x) \partial_k + b^j(x) \partial_j + c(x), \quad \exists \theta > 0 \quad a^{jk} \xi_j \xi_k \geq \theta |\xi|^2 \forall x \in \Omega, \forall \xi \in \mathbb{R}^n.$$ 

The next question is how smooth (regular) are such solutions – for example, are they $C^2(\bar{\Omega})$ and therefore solutions in the classical sense?

The natural way to measure regularity here is via Sobolev spaces and Sobolev norms. It is convenient to use multi-index notation for partial derivatives:

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), 0 \leq \alpha_j \in \mathbb{Z}, \quad \partial^{(\alpha)} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}, \quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$ 

Recall:

<table>
<thead>
<tr>
<th>for $0 \leq k \in \mathbb{Z}$, the Sobolev norm</th>
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<tr>
<td>$|f|<em>{H^k(\Omega)} : = \left( \sum</em>{</td>
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<tr>
<td>defines the Sobolev space (a Banach space)</td>
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<tr>
<td>$H^k(\Omega) : = { f : \Omega \to \mathbb{R}</td>
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The regularity question divides into two: regularity at interior points $x \in \Omega$; and regularity up to the boundary $\partial \Omega$.

Interior Regularity

A little notation:

- $L^\infty(\Omega)$ denotes the bounded functions on $\Omega$ (strictly speaking: measurable functions which are bounded except possibly on sets of zero measure) and its norm $\|f\|_{L^\infty(\Omega)} : = \sup_{x \in \Omega} |f(x)|$ (strictly speaking, supremeum after possibly removing sets of measure zero);

- $V \subset \subset \Omega$ denotes $V$ compactly contained in $\Omega$: i.e. its closure $\bar{V} \subset \Omega$. 

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The ellipticity again, and Young's inequality: 
\[ Lu = (which it is not!) to justify integrations by parts: multiply the PDE supported in \( \Omega \).

\[ \text{estimate} \]

A where we have used a notation very common in PDE analysis:

\[ \text{and so arrive at} \]

\[ \text{Ellipticity, Young's inequality, and Hölder's inequality:} \]

\[ \text{Proof (sketch):} \]

\[ \text{Remarks:} \]

\[ \text{• note that boundary conditions play no role here;} \]

\[ \text{• on first sight it might seem weird to have} \| u \|_{L^2} \text{ appear on the right-hand side,} \]

\[ \text{but since BCs play no role, imagine, eg.,} \quad L = -\Delta \text{ and } u(x) = A \text{ is a constant;} \]

\[ \text{• gain of two derivatives} f \in L^2 \implies u \in H^2 \text{ is a typical elliptic regularity result;} \]

\[ \text{• note that if} \quad u \in H^2_{\text{loc}}, \text{ then} \quad Lu \in L^2, \text{ and the original PDE holds as an equality of} \]

\[ \text{of} \]

\[ \text{L^2 \ functions – in particular, is at least satisfied “almost everywhere”}. \]

\[ \text{A} \lesssim B \text{ means} \quad A \leq CB \text{ for some constant} \quad C \text{ (which is fixed for the estimate of interest)}. \]

\[ \text{As is completely typical in PDE analysis, the main ingredient is an a priori estimate:} \]

\[ \text{suppose} \quad u \text{ is in fact smooth (which we do not know yet), and compactly supported in} \quad \Omega \text{ (which it is not!) to justify integrations by parts: multiply the PDE} \]

\[ \text{Lu = f by the second derivative} \quad u_{ll} \text{ (for some} \quad 1 \leq l \leq n), \text{ integrate by parts, and use the ellipticity again, and Young’s inequality:} \]

\[ 0 = \int_{\Omega} u_{ll} \left[ -(a^{jk} u_k)_j + b^j u_j + c u - f \right] = \int_{\Omega} \left[ u_{ll} a^{jk} u_k + b^j u_j u_{ll} + cu_{ll} - f u_{ll} \right] \]

\[ = \int_{\Omega} \left[ -a^{jk} u_j u_{lk} - u_j (a^{jk})_l u_k + b^j u_j u_{ll} + cu_{ll} - f u_{ll} \right] \]

\[ \leq -\theta \int_{\Omega} \| \nabla u \|^2 + \int_{\Omega} \left[ \frac{\theta}{2n} |D^2 u|^2 + \frac{n}{2\theta} \left( (\| \nabla a \|^2 + |b|^2) |\nabla u|^2 + \| c \|^2 u^2 + f^2 \right) \right] \]
and so summing over \( l \), and using our first estimate,
\[
\|D^2u\|_{L^2(\Omega)} \lesssim \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \lesssim \|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}.
\]
Now, here’s how we can make this estimate rigorous. Fix \( V \subset \Omega \).

1. to justify the integration by parts without boundary terms, we will multiply through by a smooth cut-off function:
\[
\eta \in C_c^\infty(\Omega), \quad \eta \equiv 1 \text{ on } V
\]
(the existence of such an \( \eta \) is a fact from analysis, related to Urysohn’s lemma);

2. to justify using \( u \) as a test function, we will replace it with a finite-difference approximation:
\[
u := -D_i^{-h} \left( \eta^2 D_h^i \right) \in H_0^1(\Omega), \quad D_h^i u(x) := \frac{u(x + h \hat{e}_i) - u(x)}{h}
\]

Inserting this \( v \) into the weak form of the PDE and “integrating by parts” as above, yields
\[
\int_V |D^h \nabla u|^2 \lesssim \|u\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2,
\]
and then taking \( h \to 0 \) yields
\[
\int_V |D^2u|^2 \lesssim \|u\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2,
\]
from which the desired estimate follows. \( \square \)

By an iteration of this argument, we can obtain higher interior regularity, as long as the source term and coefficients are sufficiently smooth:

**Theorem (higher interior regularity):** let \( m \in \mathbb{N} \). If \( a^{ij}, b^i, c \in C^{m+1}(\Omega) \) and \( f \in H^m(\Omega) \), then a (weak) solution of \( Lu = f \) satisfies \( u \in H^{m+2}_{loc}(\Omega) \) (with estimates as above).

We can relate Sobolev regularity back to classical regularity by way of the **Sobolev embedding theorem:** for \( k \in \mathbb{N} \),
\[
u \in H^m_{loc}(\Omega) \text{ for some } m > k + \frac{n}{2} \implies \nu \in C^k(\Omega).
\]
So as a corollary of the higher regularity, we have:

- \( a^{ij}, b^i, c \in C^{m+1}(\Omega), f \in H^m(\Omega) \) for some \( m > n/2 \implies u \in C^2(\Omega) \) is a classical solution of \( Lu = f \);
- \( a^{ij}, b^i, c \in C^\infty(\Omega), f \in C^\infty(\Omega) \implies u \in C^\infty(\Omega) \) (this generalizes \( u \) harmonic \( \implies u \in C^\infty(\Omega) \)).