The University of British Columbia

Final Examination - December 12, 2012

Mathematics 420/507

Closed book examination Time: 2.5 hours

Last Name ____________ **First** ________ **Signature** ____________

Student Number ____________

Special Instructions:

No books, notes, or calculators are allowed.

1		17
2		17
3		17
4		17
5		16
6		16
Total		100

1. [17] Let X be a set, $\mathcal{M} \subset \mathcal{P}(X)$ a σ-algebra, and $\mathcal{A} \subset \mathcal{P}(X)$ an algebra.

   (a) Give definitions of:

      i. an outer measure on X

      ii. a measure on $(X, \mathcal{M})$

      iii. a premeasure on $(X, \mathcal{A})$

   (b) Let ρ be a premeasure on $\mathcal{A}$. Explain briefly (no proofs!) how it gives rise to a measure.

   (c) Let μ be a σ-finite measure on $(X, \mathcal{M})$, where $\mathcal{M}$ is the σ-algebra generated by an algebra $\mathcal{A}$. Given $E \in \mathcal{M}$ with $\mu(E) < \infty$, and $\epsilon > 0$, show there is $A \in \mathcal{A}$ such that $E \subset A$, and $\mu(A \backslash E) < \epsilon$.

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2. [17] Let A be the set of rational numbers in $[0, 1]$, and let m denote Lebesgue measure on $\mathbb{R}$.

   (a) Show that for any $\epsilon > 0$, there is a countable union of open intervals $U = \cup_{j=1}^{\infty} I_J$ such that $A \subset U$ and $m(U) < \epsilon$.

   (b) Show that A is Lebesgue measurable and find $m(A)$.

   (c) If $U = \cup_{j=1}^{N} I_j$ is a *finite* union of open intervals such that $A \subset U$, show that $m(U) > 1$.

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3. [17] Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (a) State Fatou's Lemma, and the Dominated Convergence Theorem.

   (b) Suppose $f \in L^1(\mathbb{R}, \mathcal{L}, m)$. Show that the function $F(x) = \int_{-\infty}^{x} f(t)\, dt$ is continuous.

   (c) Prove the following variant of the Dominated Convergence Theorem: Suppose $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$, f, and g are functions in $L^1(\mu)$ such that $f_n \to f$ and $g_n \to g$ (pointwise), $|f_n| \leq g_n$, and $\int g_n\, d\mu \to \int g\, d\mu$. Show that $\int f_n\, d\mu \to \int f\, d\mu$.

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4. [17] Let $(X, \mathcal{M}, \mu)$ be a measure space, and $\{f_n\}$ a sequence of measurable functions converging a.e. to a measurable function f. For each statement: if true, give a brief proof; if false, give a counterexample.

   (a) $f_n \to f$ in measure

   (b) if $\mu(X) < \infty$, $f_n \to f$ in measure

   (c) if $\mu(X) < \infty$, $f_n \to f$ in L^1

   (d) if $f_n \to 0$ in L^p for all $1 \leq p < 2$, then $f_n \to 0$ in L^2

   (e) if $\mu(X) < \infty$ and $f_n \to 0$ in L^p for some $p > 2$, then $f_n \to 0$ in L^2

   (f) if $L^1 \ni f_n \geq 0$ and $\int f_n \to \int f < \infty$, then $f_n \to f$ in L^1

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5. [16]

(a) Let $X = \mathbb{N} = \{1, 2, 3, \ldots\}$ (the natural numbers) and on $\mathcal{M} = \mathcal{P}(X)$, define

- $\mu(E) = \sum_{n \in E} \sin^2(\pi n/2)$ (a positive measure)
- $\nu(E) = \sum_{n \in E} (-1)^n/n^2$ (a signed measure)

Find the Lebesgue-Radon-Nikodym decomposition of ν with respect to μ.

(b) Suppose μ and λ are finite (positive) measures on a measurable space $(X, \mathcal{M})$, and define the signed measure $\nu = \mu - \lambda$. Show that $|\nu| \leq \mu + \lambda$.

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6. [16] Let $f \in L^1(\mathbb{R}, m)$, and set $F(x) := \int_{-\infty}^{x} f(t)\, dt$.

(a) Recall that $F : \mathbb{R} \to \mathbb{C}$ is called *Lipshitz* if for some constant K, $|F(y) - F(x)| \leq K|y - x|$ for all $x, y \in \mathbb{R}$. Prove that if $f \in L^\infty(\mathbb{R}, m)$, then F is Lipshitz.

(b) $F : \mathbb{R} \to \mathbb{C}$ is called *Hölder continuous* (with exponent $\alpha \in (0, 1)$) at x, if for some constant K, $|F(y) - F(x)| \leq K|y - x|^\alpha$ for all y near x. Prove that if $f \in L^p(\mathbb{R}, m)$ for some $p \in (1, \infty)$, then F is Hölder continuous at every $x \in \mathbb{R}$ (and find the largest exponent α for which this holds).

(c) Recall that $F : \mathbb{R} \to \mathbb{C}$ is called *absolutely continuous* if for any $\epsilon > 0$ there is $\delta > 0$ such that if $(a_1, b_1), \ldots, (a_N, b_N)$ are disjoint intervals with $\sum_{j=1}^{N}(b_j - a_j) < \delta$, then $\sum_{j=1}^{N} |F(b_j) - F(a_j)| < \epsilon$. Prove that if $f \in L^1(\mathbb{R}, m)$, then F is absolutely continuous.

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