

The University of British Columbia

Final Examination - December 12, 2012

Mathematics 420/507

Closed book examination

Time: 2.5 hours

Last Name \_\_\_\_\_ First \_\_\_\_\_ Signature \_\_\_\_\_

Student Number \_\_\_\_\_

Special Instructions:

No books, notes, or calculators are allowed.

**Rules governing examinations**

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCCard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (a) speaking or communicating with other candidates, unless otherwise authorized;
  - (b) purposely exposing written papers to the view of other candidates or imaging devices;
  - (c) purposely viewing the written papers of other candidates;
  - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		17
2		17
3		17
4		17
5		16
6		16
Total		100

1. [17] Let  $X$  be a set,  $\mathcal{M} \subset \mathcal{P}(X)$  a  $\sigma$ -algebra, and  $\mathcal{A} \subset \mathcal{P}(X)$  an algebra.
  - (a) Give definitions of:
    - i. an outer measure on  $X$
    - ii. a measure on  $(X, \mathcal{M})$
    - iii. a premeasure on  $(X, \mathcal{A})$
  - (b) Let  $\rho$  be a premeasure on  $\mathcal{A}$ . Explain briefly (no proofs!) how it gives rise to a measure.
  - (c) Let  $\mu$  be a  $\sigma$ -finite measure on  $(X, \mathcal{M})$ , where  $\mathcal{M}$  is the  $\sigma$ -algebra generated by an algebra  $\mathcal{A}$ . Given  $E \in \mathcal{M}$  with  $\mu(E) < \infty$ , and  $\epsilon > 0$ , show there is  $A \in \mathcal{A}$  such that  $E \subset A$ , and  $\mu(A \setminus E) < \epsilon$ .

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2. [17] Let  $A$  be the set of rational numbers in  $[0, 1]$ , and let  $m$  denote Lebesgue measure on  $\mathbb{R}$ .
- (a) Show that for any  $\epsilon > 0$ , there is a countable union of open intervals  $U = \cup_{j=1}^{\infty} I_j$  such that  $A \subset U$  and  $m(U) < \epsilon$ .
  - (b) Show that  $A$  is Lebesgue measurable and find  $m(A)$ .
  - (c) If  $U = \cup_{j=1}^N I_j$  is a *finite* union of open intervals such that  $A \subset U$ , show that  $m(U) > 1$ .

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3. [17] Let  $(X, \mathcal{M}, \mu)$  be a measure space.
- (a) State Fatou's Lemma, and the Dominated Convergence Theorem.
  - (b) Suppose  $f \in L^1(\mathbb{R}, \mathcal{L}, m)$ . Show that the function  $F(x) = \int_{-\infty}^x f(t) dt$  is continuous.
  - (c) Prove the following variant of the Dominated Convergence Theorem: Suppose  $\{f_n\}_{n=1}^{\infty}$ ,  $\{g_n\}_{n=1}^{\infty}$ ,  $f$ , and  $g$  are functions in  $L^1(\mu)$  such that  $f_n \rightarrow f$  and  $g_n \rightarrow g$  (pointwise),  $|f_n| \leq g_n$ , and  $\int g_n d\mu \rightarrow \int g d\mu$ . Show that  $\int f_n d\mu \rightarrow \int f d\mu$ .

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4. [17] Let  $(X, \mathcal{M}, \mu)$  be a measure space, and  $\{f_n\}$  a sequence of measurable functions converging a.e. to a measurable function  $f$ . For each statement: if true, give a brief proof; if false, give a counterexample.
- (a)  $f_n \rightarrow f$  in measure
  - (b) if  $\mu(X) < \infty$ ,  $f_n \rightarrow f$  in measure
  - (c) if  $\mu(X) < \infty$ ,  $f_n \rightarrow f$  in  $L^1$
  - (d) if  $f_n \rightarrow 0$  in  $L^p$  for all  $1 \leq p < 2$ , then  $f_n \rightarrow 0$  in  $L^2$
  - (e) if  $\mu(X) < \infty$  and  $f_n \rightarrow 0$  in  $L^p$  for some  $p > 2$ , then  $f_n \rightarrow 0$  in  $L^2$
  - (f) if  $L^1 \ni f_n \geq 0$  and  $\int f_n \rightarrow \int f < \infty$ , then  $f_n \rightarrow f$  in  $L^1$



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5. [16]

(a) Let  $X = \mathbb{N} = \{1, 2, 3, \dots\}$  (the natural numbers) and on  $\mathcal{M} = \mathcal{P}(X)$ , define

- $\mu(E) = \sum_{n \in E} \sin^2(\pi n/2)$  (a positive measure)
- $\nu(E) = \sum_{n \in E} (-1)^n/n^2$  (a signed measure)

Find the Lebesgue-Radon-Nikodym decomposition of  $\nu$  with respect to  $\mu$ .

(b) Suppose  $\mu$  and  $\lambda$  are finite (positive) measures on a measurable space  $(X, \mathcal{M})$ , and define the signed measure  $\nu = \mu - \lambda$ . Show that  $|\nu| \leq \mu + \lambda$ .

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6. [16] Let  $f \in L^1(\mathbb{R}, m)$ , and set  $F(x) := \int_{-\infty}^x f(t) dt$ .
- (a) Recall that  $F : \mathbb{R} \rightarrow \mathbb{C}$  is called *Lipshitz* if for some constant  $K$ ,  $|F(y) - F(x)| \leq K|y - x|$  for all  $x, y \in \mathbb{R}$ . Prove that if  $f \in L^\infty(\mathbb{R}, m)$ , then  $F$  is Lipshitz.
  - (b)  $F : \mathbb{R} \rightarrow \mathbb{C}$  is called *Hölder continuous* (with exponent  $\alpha \in (0, 1)$ ) at  $x$ , if for some constant  $K$ ,  $|F(y) - F(x)| \leq K|y - x|^\alpha$  for all  $y$  near  $x$ . Prove that if  $f \in L^p(\mathbb{R}, m)$  for some  $p \in (1, \infty)$ , then  $F$  is Hölder continuous at every  $x \in \mathbb{R}$  (and find the largest exponent  $\alpha$  for which this holds).
  - (c) Recall that  $F : \mathbb{R} \rightarrow \mathbb{C}$  is called *absolutely continuous* if for any  $\epsilon > 0$  there is  $\delta > 0$  such that if  $(a_1, b_1), \dots, (a_N, b_N)$  are disjoint intervals with  $\sum_{j=1}^N (b_j - a_j) < \delta$ , then  $\sum_{j=1}^N |F(b_j) - F(a_j)| < \epsilon$ . Prove that if  $f \in L^1(\mathbb{R}, m)$ , then  $F$  is absolutely continuous.

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