Math 420/507: Assignment 6 (Due Friday, Nov. 30)

Unless otherwise noted, you may use any result from Chapters 0, 1, 2, or 3 of Folland.

1. Find the Lebesgue set L_f, if:
   (a) $f : \mathbb{R}^n \to \mathbb{C}$ is continuous;
   (b) $f = \chi_E$, $E \in \mathcal{B}_{\mathbb{R}^n}$ (Lebesgue) null: $m(E) = 0$;
   (c) $f(x) = \lfloor x \rfloor$ on $\mathbb{R}$.

2. (a) If $F : \mathbb{R} \to \mathbb{C} \in$ NBV, show that there is a (Lebesgue) null Borel set $N \subset \mathbb{R}$ such that if $[a, b] \cap N = \emptyset$, then
$$F(b) - F(a) = \int_a^b F'(t)dt.$$
   (b) Show that if $F : \mathbb{R} \to \mathbb{R}$ is increasing, and $-\infty < a < b < \infty$,
$$F(b) - F(a) \geq \int_a^b F'(t)dt.$$

3. For the function $F(x) = |x|$ on $[-1, 1]$:
   (a) show *directly from the definition* that F is absolutely continuous on $[-1, 1]$;
   (b) show *directly* that F satisfies the FTC: $F(x) - F(-1) = \int_{-1}^x F'(t)dt$.

4. On $(-1, 1)$, $F_1(x) = x^2 \sin(\frac{1}{x})$, $F_2(x) = x^2 \sin(\frac{1}{x^2})$, $F_3(x) = x^2 \sin(\frac{1}{x^{\frac{4}{3}}})$ ($F_j(0) = 0$):
   (a) Show that each F_j is (everywhere) differentiable.
   (b) Show that F_1 is absolutely continuous (hence also BV). *Hint: show* $|F_1'|$ *bounded.*
   (c) Show F_2 is not BV (hence also not AC). *Hint: consider* F_2 *at* $x_n = (\pi/2 + n\pi)^{-1/2}$.
   (d) Show F_3 is absolutely continuous.

(Nov. 22)