Math 420/507: Assignment 5 Solutions

Unless otherwise noted, you may use any result from Ch. 0, 1, 2 or 3.1 – 3.4 of Folland.

1. For \( x \in \mathbb{R} \), let \( |x| \) denote the greatest integer less than or equal to \( x \), and define \( F : \mathbb{R} \to \mathbb{R} \) by \( F(x) := |x||x| \).

   (a) Verify that \( F \) is non-decreasing and right continuous.

   On each interval \([k, k+1)\), \( k \in \mathbb{Z} \), \( F(x) = k|x| \) is continuous and non-decreasing \((F' = |k| \geq 0)\). Moreover, at each integer \( k \) \( F(k^+) = F(k) = |k| \) and the jump is \( F(k^+) - F(k^-) = |k||k| - (k-1)|k| = |k| \geq 0 \). Hence \( f \) is non-decreasing and right continuous.

   (b) Find the Lebesgue-Radon-Nikodym decomposition of the corresponding Lebesgue-Stieltjes measure \( \mu_F \) on \( \mathbb{R} \), with respect to Lebesgue measure.

   Set \( \lambda := \sum_{k \in \mathbb{Z}} |k|\delta_k \).

   Since the set \( \mathbb{Z} \) is null for \( m \) (since countable), and the set \( \mathbb{R}\setminus\mathbb{Z} \) is null for \( \lambda \), we see that \( \lambda \perp m \). Next set \( f(x) := |\lfloor x \rfloor| \) and note that \( f \in L^+(m) \) (it is measurable since it is continuous a.e.), and define the measure \( d\rho := d\lambda+f\,dm \).

   Now consider an interval \((a,b)\), with \( 0 < b - a < 1 \). If this interval contains no integer, then \( [a] = [b] \) and

   \[
   \rho((a,b)) = \int_a^b |\lfloor x \rfloor| \, dx = \int_a^b |[a]| \, dx = |[a]|(b-a) = |b|b-|[a]|a
   \]

   while if integer \( k \in (a,b) \), then \( [a] = k - 1 \), \( [b] = k \), and

   \[
   \rho((a,b)) = |k| + \int_a^k |k-1| \, dx + \int_k^b |k| \, dx = |k| + |k-1|(k-a) + |k|(b-k)
   = |k|b-|k-1|a = |b|b-|a|a = |b|b-|a|a = F(b) - F(a) = \mu_F((a,b))
   \]

   Then since any interval of the form \((a,b)\) is a finite union of such intervals, we see that \( \rho \) and \( \mu_F \) agree on intervals \((a,b)\). Since \( \mu_F \) is the unique Borel measure assigning this measure to such intervals, we see \( \rho \) and \( \mu_F \) agree on Borel sets. Thus

   \[
   d\mu_F = d\left(\sum_{k \in \mathbb{Z}} |k|\delta_k\right) + |\lfloor x \rfloor| dm(x)
   \]

   is the Lebesgue-Radon-Nikodym decomposition of \( \mu_F \) with respect to Lebesgue measure (on Borel sets).

2. (Folland 3.2 # 13). Set \( X = [0, 1] \), \( \mathcal{M} = \mathcal{B}_{[0,1]} \), \( m = \) Lebesgue measure on \( \mathcal{M} \), \( \mu = \) counting measure on \( \mathcal{M} \).
(a) Show that \( m \ll \mu \), but that \( dm \not= f \, \text{d}\mu \) for any \( f \).

If \( \mu(E) = 0 \) then \( E \) is a finite set, so \( m(E) = 0 \). Thus \( m \ll \mu \). However, if \( dm = f \, \text{d}\mu \) for some \( f \geq 0 \), then

\[
1 = m(X) = \int_X f \, \text{d}\mu = \sum_{x \in X} f(x),
\]

which implies \( E := \{ x \in X \mid f(x) > 0 \} \) is countable. Thus

\[
0 = \int_{X \setminus E} f \, \text{d}\mu = m(X \setminus E) = m(X) - m(E) = 1 - 0 = 1,
\]
a contradiction.

(b) Show that \( \mu \) has no Lebesgue decomposition with respect to \( m \).

Suppose \( \mu = \lambda + \rho \) with \( \lambda \perp m \), \( \rho \ll m \). Then \( X = E \cup F \) (disjoint union) with \( E \) \( m \)-null and \( F \) \( \lambda \)-null. Thus \( m(F) = m(X) - m(E) = 1 - 0 = 1 \), and so in particular \( F \) is infinite. Let \( A \) be a countably infinite subset of \( F \). Since \( \mu \) and \( \rho \) agree on \( F \), \( \rho(A) = \mu(A) = \infty \). But since \( \rho \ll m \) and \( m(A) = 0 \), we have \( \rho(A) = 0 \), a contradiction.

(c) Why don’t the Lebesgue decomposition and Radon-Nikodym theorems apply?

Because \( \mu \) is not \( \sigma \)-finite.

3. (c.f. Folland 3.2 \# 17). Let \((X, \mathcal{M}, \mu)\) be a finite measure space, let \( \mathcal{N} \subset \mathcal{M} \) be a sub-\( \sigma \)-algebra, and let \( \nu = \mu \mid \mathcal{N} \).

(a) If \( f \in L^1(\mu) \), show there exists \( g \in L^1(\nu) \) such that \( \int_E f \, \text{d}\mu = \int_E g \, \text{d}\nu \) for all \( E \in \mathcal{N} \), which is (\( \nu \) a.e.) unique. Remark: in probability, \( g \) is the conditional expectation of \( f \) on \( \mathcal{N} \).

Consider the complex measure \( \text{d}\lambda = f \, \text{d}\mu \) on \( \mathcal{N} \). Clearly \( \lambda \ll \nu \), so the Radon-Nikodym theorem provides \( g = \text{d}\lambda/\text{d}\nu \in L^1(\nu) \) (in particular \( \mathcal{N} \)-measurable) s.t. for all \( E \in \mathcal{N} \), \( \lambda(E) = \int_E f \, \text{d}\mu = \int_E g \, \text{d}\nu \), and also gives us uniqueness of \( g \) with this property (up to \( \nu \)-null sets).

(b) Determine \( g \) (in terms of \( f \)) in the cases \( \mathcal{N} = \{ \emptyset, X \} \) and \( \mathcal{N} = \{ \emptyset, E^c, X \} \) (for some \( E \in \mathcal{M} \)). Hint: first determine which functions are measurable with respect to these \( \sigma \)-algebras.

For \( \mathcal{N} = \{ \emptyset, X \} \), the only \( \mathcal{N} \)-measurable functions are constants, so \( g(x) = C \), and \( C \) is determined by \( \int_X f \, \text{d}\mu = \int_X C \, \text{d}\nu = C \nu(X) = C \mu(X) \) (recall this is a finite measure space). So \( g(x) \equiv C \frac{\mu(X)}{\nu(X)} \int_X f \, \text{d}\mu \). For \( \mathcal{N} = \{ \emptyset, E^c, X \} \), the \( \mathcal{N} \)-measurable functions are \( g = c_1 \chi_E + c_2 \chi_{E^c} \), and the coefficients are determined by the relations \( \int_E f \, \text{d}\mu = c_1 \mu(E) \) and \( \int_{E^c} \text{d}\mu = c_2 \mu(E^c) \), so

\[
g(x) = \frac{\int_E f \, \text{d}\mu}{\mu(E)} \chi_E(x) + \frac{\int_{E^c} f \, \text{d}\mu}{\mu(E^c)} \chi_{E^c}(x).
\]

4. On \( (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, m) \), let \( Hf \) denote the maximal function of \( f \in L^1_{\text{loc}} \).
(a) If \( f \in L^1 \), show that \( m(\{x \mid |f(x)| > \alpha \}) \leq \frac{1}{\alpha} \int |f| \)

Remark: this is a “weak \( L^1 \)-type estimate”.

\[
\int |f| \geq \int_{\{f > \alpha\}} \geq \alpha \int_{\{f > \alpha\}} 1 = \alpha m(\{f > \alpha\}),
\]

and the result follows.

(b) If \( f \in L^1 \) and \( f \) is not a.e. \( 0 \), show that for some \( C > 0 \), \( Hf(x) \geq C/|x|^n \) for all \( x \) with \( |x| \geq 1 \). Conclude \( Hf \notin L^1 \).

Remark: this despite that fact that by the Maximal Theorem, a weak \( L^1 \)-type estimate \( m(\{x \mid Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \) holds.

Since \( f \) is not a.e. \( 0 \), we must have \( 0 < \int_{B(R,0)} |f| =: M \) for some sufficiently large \( R \). Then for any \( |x| \geq R \), \( B(|x| + R, x) \supset B(R,0) \), and so

\[
Hf(x) \geq \frac{1}{m(B(|x| + R, x))} \int_{B(|x| + R, x)} |f| \geq \frac{c}{|x|^n + R^n} \int_{B(R,0)} |f| \geq \frac{cM}{|x|^n + R^n},
\]

and the desired bound follows from noting that if \( |x| \geq 1 \), then \( R \leq R|x| \) and so \( |x|^n + R^n \leq (1 + R^n)|x|^n \).

Then working in polar coordinates, for any \( N \geq 1 \),

\[
\int_{1 \leq |x| \leq N} Hf(x) \geq C \int_{1 \leq |x| \leq N} \frac{1}{|x|} = C \int_1^N \frac{1}{r} dr = C' \log(N) \to \infty
\]

as \( N \to \infty \), so \( Hf \notin L^1 \).

5. On \((\mathbb{R}, \mathcal{B}_\mathbb{R}, m)\), for \( f \in L^1_{\text{loc}}(m) \), set \( F(x) = \int_0^x f(t)dt \). Recall \( A_r f(x) = \frac{1}{2r} \int_{x-r}^{x+r} f(t)dt \).

(a) Show that \( F \) is continuous.

Let \( \epsilon > 0 \). Since \( f \in L^1 \), we know there is \( \delta > 0 \) such that \( m(E) < \delta \implies \int_E |f| < \epsilon \). Then if \( |y - x| < \delta \), \(|F(y) - F(x)| = |f_y| f(t)dt| < \epsilon \). Thus \( F \) is (in fact uniformly) continuous.

(b) Show that if \( F \) is differentiable at \( 0 \), then \( \lim_{r \to 0^+} A_r f(0) = F'(0) \).

Since \( F(0) = 0 \),

\[
\lim_{r \to 0^+} \frac{1}{r} \int_0^r f(t)dt = \lim_{r \to 0^+} \frac{F(r)}{r} = F'(0) = \lim_{r \to 0^+} \frac{F(-r)}{-r} = \lim_{r \to 0^+} \frac{1}{r} \int_{-r}^0 f(t)dt,
\]

so

\[
\lim_{r \to 0^+} A_r f(0) = \lim_{r \to 0^+} \frac{1}{2r} \int_{x-r}^{x+r} f(t)dt = \frac{1}{r} \lim_{r \to 0^+} \left[ \frac{1}{r} \int_0^r f(t)dt + \frac{1}{r} \int_{-r}^0 f(t)dt \right] = F'(0).
\]

(c) Give an example with \( \lim_{r \to 0^+} A_r f(0) = f(0) \) but \( F \) not differentiable at \( 0 \).

Set \( f(t) = \begin{cases} -1 & t < 0 \\ 0 & t = 0 \\ 1 & t > 0 \end{cases} \). Then \( F(x) = |x| \) is not differentiable at \( x = 0 \), but \( A_r f(0) = \frac{1}{r} \int_{-r}^r f(t)dt = 0 = f(0) \) for all \( r > 0 \).
(d) Give an example with \( F \) differentiable at 0, but \( \lim_{r \to 0^+} A_r |f - F'(0)(0)| \neq 0. \)

Set \( f(t) = \sin(1/t) \) for \( t \neq 0 \). Then for \( r > 0 \),

\[
\frac{F(r)}{r} = \frac{1}{r} \int_0^r \sin \left( \frac{1}{t} \right) dt = \frac{1}{r} \int_{\frac{1}{r}}^\infty \sin(y) \frac{dy}{y^2}
\]

\[
= \frac{1}{r} \left[ -\cos(y) \frac{1}{y^2} \bigg|_{\frac{1}{r}}^\infty - 2 \int_{\frac{1}{r}}^\infty \cos(y) \frac{dy}{y^3} \right] = r \cos \left( \frac{1}{r} \right) - 2 \int_{\frac{1}{r}}^\infty \cos(y) \frac{dy}{y^3}
\]

by a change of variable, and an integration by parts. So

\[
\left| \frac{F(r)}{r} \right| \leq r + \frac{2}{r} \int_{\frac{1}{r}}^\infty \frac{dy}{y^3} \leq 2r \to 0 \text{ as } r \to 0^+,
\]

and a similar argument shows \( \lim_{r \to 0^+} \frac{F(-r)}{r} = 0 \), so \( F \) is differentiable at 0 with \( F'(0) = 0 \). However,

\[
A_r |f| (0) = \frac{1}{2r} \int_{-r}^r |\sin(1/t)| dt = \frac{1}{r} \int_0^r |\sin(1/t)| dt = \frac{1}{r} \int_{\frac{1}{r}}^\infty |\sin(y)| \frac{dy}{y^2}.
\]

For \( k \in \mathbb{N} \),

\[
\int_{k \pi}^{(k+1) \pi} |\sin(y)| \frac{dy}{y^2} \geq \int_{(k+\frac{1}{2}) \pi}^{(k+\frac{3}{2}) \pi} |\sin(y)| \frac{dy}{y^2} \geq \int_{(k+\frac{1}{2}) \pi}^{(k+\frac{3}{2}) \pi} \frac{1}{\sqrt{2} (2k)^2 \pi^2} dy \geq \frac{\pi}{2 \sqrt{2} (2k)^2 \pi^2} = C \frac{1}{k^2}
\]

so

\[
\int_{k \pi}^{\infty} |\sin(y)| \frac{dy}{y^2} \geq C \sum_{j=k}^{\infty} \frac{1}{j^2} \geq C \int_k^{\infty} \frac{dz}{z^2} = C \frac{1}{k},
\]

and so

\[
A_{\frac{1}{k \pi}} |f| (0) \geq k \pi C \frac{1}{k} = C \pi \to 0 \text{ as } k \to \infty.
\]

(Nov. 12)