Math 420/507: Assignment 4 (Due Wednesday, Nov. 7)

Unless otherwise noted, you may use any result from Chapters 0, 1, or 2 of Folland, or established in class.

1. (Riemann-Lebesgue Lemma). If $f \in L^1(dx)$, show that

$$\lim_{n \to \infty} \int_{\mathbb{R}} e^{inx} f(x) dx = 0.$$ 

(Hint: one way to proceed is to first show this when $f$ is the characteristic function of an interval.)

2. Suppose that $f_n$ is a sequence of measurable functions converging to measurable $f$ “almost uniformly” (as in Egoroff’s Theorem): that is, for any $\epsilon > 0$, there is $E \subset X$ with $\mu(E) < \epsilon$ such that $f_n \to f$ uniformly on $E^c$.

(a) Show that $f_n \to f$ a.e.

(b) Show that $f_n \to f$ in measure.

3. If $\{f\}_{n=1}^{\infty}$ is a sequence of (measurable) functions converging in measure to a (measurable) function $f$, with $|f_n(x)| \leq g(x)$ for some $g \in L^1$, show $f_n \to f$ in $L^1$.

4. Let $X = Y = \mathbb{N}$ (natural numbers) and $\mu = \nu$ be counting measure (with measurable sets $\mathcal{P}(\mathbb{N})$). Define $f : X \times Y \to \mathbb{R}$ by $f(n,m) = \delta_{n,m} - \delta_{n,m+1}$. Compute $\iint f \, d\mu \, d\nu$ and explain any discrepancy with the conclusions of Fubini’s and Tonelli’s theorems.

5. Suppose $f$ and $g$ are (Lebesgue) integrable functions on $\mathbb{R}$. The convolution of $f$ and $g$ is the function $f * g$ defined by

$$(f * g)(x) := \int_{\mathbb{R}} f(x - y) g(y) \, dy.$$ 

Show that $f * g$ is defined for a.e. $x$, measurable, and integrable (in short: $f * g \in L^1(\mathbb{R})$). Hint: apply Tonelli-Fubini to $F(x,y) := f(x-y)g(y)$.

6. Let $X = [0,1) \times [0,1) \subset \mathbb{R}^2$, and for fixed $\alpha \in \mathbb{R}$, let $E \subset X$ be the range of the map $t \mapsto (t \mod 1, \alpha t \mod 1)$, $t \geq 0$.

Show that $E$ is (Borel) measurable, and find $m^2(E)$.

7. Let $\mu$ be Lebesgue measure on $[0,1]$ and let $\nu$ be counting measure on $[0,1]$. Let $D := \{(x,x) \mid x \in [0,1]\} \subset [0,1] \times [0,1]$. Show that $D \in \mathcal{B}_{[0,1]} \otimes \mathcal{P}([0,1])$, compute $\int \nu(D_x) d\mu(x)$, $\int \mu(D_y) d\nu(y)$, and explain your results in light of Tonelli’s theorem (or equivalently the “slicing” theorem).

8. Show the following properties for signed measures $\nu$, $\mu$, $\nu_1$, $\nu_2$:

(a) $E$ is $\nu$-null iff $|\nu|(E) = 0$

(b) $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$

(c) $\nu_j \ll \mu$, $j = 1, 2 \implies \nu_1 + \nu_2 \ll \mu$

(d) $\nu_j \perp \mu$, $j = 1, 2 \implies \nu_1 + \nu_2 \perp \mu$

(Oct. 22)