Math 420/507: Assignment 4 (Due Wednesday, Nov. 7)

Unless otherwise noted, you may use any result from Chapters 0, 1, or 2 of Folland, or established in class.

1. (*Riemann-Lebesgue Lemma*). If $f \in L^1(dx)$, show that
$$\lim_{n\to\infty} \int_{\mathbb{R}} e^{inx} f(x)dx = 0.$$
(*Hint: one way to proceed is to first show this when f is the characteristic function of an interval.*)

2. Suppose that f_n is a sequence of measurable functions converging to measurable f "almost uniformly" (as in Egoroff's Theorem): that is, for any $\epsilon > 0$, there is $E \subset X$ with $\mu(E) < \epsilon$ such that $f_n \to f$ uniformly on E^c.
   (a) Show that $f_n \to f$ a.e.
   (b) Show that $f_n \to f$ in measure.

3. If $\{f\}_{n=1}^{\infty}$ is a sequence of (measurable) functions converging in measure to a (measurable) function f, with $|f_n(x)| \le g(x)$ for some $g \in L^1$, show $f_n \to f$ in L^1.

4. Let $X = Y = \mathbb{N}$ (natural numbers) and $\mu = \nu$ be counting measure (with measurable sets $\mathcal{P}(\mathbb{N})$). Define $f : X \times Y \to \mathbb{R}$ by $f(n,m) = \delta_{n,m} - \delta_{n,m+1}$. Compute $\iint f\, d\mu\, d\nu$ and $\iint f\, d\nu\, d\mu$, and explain any discrepancy with the conclusions of Fubini's and Tonelli's theorems.

5. Suppose f and g are (Lebesgue) integrable functions on $\mathbb{R}$. The convolution of f and g is the function $f * g$ defined by
$$(f * g)(x) := \int_{\mathbb{R}} f(x-y)g(y)\, dy.$$
Show that $f * g$ is defined for a.e. x, measurable, and integrable (in short: $f * g \in L^1(\mathbb{R})$). *Hint: apply Tonelli-Fubini to $F(x,y) := f(x-y)g(y)$.*

6. Let $X = [0,1) \times [0,1) \subset \mathbb{R}^2$, and for fixed $\alpha \in \mathbb{R}$, let $E \subset X$ be the range of the map
$$t \mapsto (t \bmod 1,\ \alpha t \bmod 1), \quad t \ge 0.$$
Show that E is (Borel) measurable, and find $m^2(E)$.

7. Let μ be Lebesgue measure on $[0,1]$ and let ν be counting measure on $[0,1]$. Let $D := \{(x,x) \mid x \in [0,1]\} \subset [0,1] \times [0,1]$. Show that $D \in \mathcal{B}_{[0,1]} \otimes \mathcal{P}([0,1])$, compute $\int \nu(D_x)d\mu(x)$, $\int \mu(D^y)d\nu(y)$, abd $(\mu \times \nu)(D)$, and explain your results in light of Tonelli's theorem (or equivalently the "slicing" theorem).

8. Show the following properties for signed measures ν, μ, ν_1, ν_2:
   (a) E is ν-null iff $|\nu|(E) = 0$
   (b) $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$
   (c) $\nu_j \ll \mu$, $j = 1,2 \implies \nu_1 + \nu_2 \ll \mu$
   (d) $\nu_j \perp \mu$, $j = 1,2 \implies \nu_1 + \nu_2 \perp \mu$

(Oct. 22)