Math 420/507: Assignment 3 (Due Wednesday, Oct. 24)

Unless otherwise noted, you may use any result from Chapters 0, 1, or 2.1-2.4 of Folland, or established in class.

1. For some $p \geq 0$, consider the sequence of functions $f_n(x) = n^p \chi_0(x/n)$ on $\mathbb{R}$.

   (a) Show that $f_n \to 0$ pointwise.

   For any $x > 0$, $f_n(x) = 0$ for all $n > \frac{1}{x}$, while for any $x \leq 0$, $f_n(x) = 0$ for all $n$. Thus $f_n(x) \to 0$.

   (b) For which $p$ is $\lim_{n \to \infty} \int f_n \, dm = 0$?

   $$\int f_n = n^p \cdot \frac{1}{n} = n^{p-1} \to \begin{cases} 0 & p < 1 \\ 1 & p = 1 \\ \infty & p > 1 \end{cases}$$

   (c) Find a dominating function $g$ such that $f_n(x) \leq g(x)$ for all $n$, and interpret the result of (b) in light of $g$ and the dominated convergence theorem.

   $n^p \chi_0(x/n) \leq g(x) = x^{-p} \chi_{(0,1)}(x)$. For $p < 1$, $g \in L^1$, and the DCT applies to ensure $\lim_n \int f_n = \int \lim_n f_n = 0$. While for $p \leq 1$, $g \notin L^1$, and DCT does not apply.

2. In each case, identify $L^1(\mu)$, and compute $\int f \, d\mu$ for $f \in L^1(\mu)$:

   (a) on $\mathbb{N}$, $\mu = \text{counting measure}$.

   Any $f : \mathbb{N} \to \mathbb{C}$ can be written as $f = \sum_{n=1}^\infty f(n) \chi_{\{n\}}$, and so $|f| = \sum_{n=1}^\infty |f(n)| \chi_{\{n\}}$.

   Since the integral of the simple function $\chi_{\{n\}}$ is $\mu(\{n\}) = 1$, we have (by countable additivity) $\int |f| \, d\mu = \sum_{n=1}^\infty |f(n)|$, and

   $$f \in L^1 \iff \sum_{n=1}^\infty |f(n)| < \infty, \quad \int f \, d\mu = \sum_{n=1}^\infty f(n).$$

   (b) on $\mathbb{R}$, $\mu = \text{counting measure}$.

   Suppose $f \in L^1$. Then for any $\delta > 0$, $\mu(\{|f(x)| > \delta\}) < \infty$ (otherwise $\int |f| \, d\mu \geq \int_{\{|f(x)| > \delta\}} |f| \geq \delta \mu(\{|f(x)| > \delta\}) = \infty$), and so this set is finite. Since $\{f(x) \neq 0\} = \bigcup_{k=1}^{\infty} \{|f(x)| \geq \frac{1}{k}\}$ is a countable union of finite sets, it is countable, and so expressible as $\{x_1, x_2, x_3, \ldots\}$. Then $f = \sum_{j=1}^{\infty} f(x_j) \chi_{\{x_j\}}$. So as in part (a),

   $$f \in L^1 \iff \{f(x) \neq 0\} \text{ is countable, and } \sum_{\{x \mid f(x) \neq 0\}} |f(x)| < \infty,$$

   and in this case $\int f \, d\mu = \sum_{\{x \mid f(x) \neq 0\}} f(x)$.

   (c) on $\mathbb{R}$, $\mu = \mu_F$, the Lebesgue-Stieltjes measure with $F(x) = |x|$;

   Recall $m_F = \sum_{j \in \mathbb{Z}} \delta_j$. So for a simple function $f = \sum_{k=1}^n z_k \chi_{E_k}$, $z_k \geq 0$,

   $$\int f \, dm_F = \sum_{k=1}^n z_k \#(\mathbb{Z} \cap E_k) = \sum_{j \in \mathbb{Z}} f(j).$$
So if $0 \leq \phi_n^{\text{simple}} \leq |f|$, with $\phi_n \uparrow |f|$ pointwise (we know such a sequence of simple functions exists), by the monotone convergence theorem,

$$
\int |f| \, dm_F = \lim_{n \to \infty} \int \phi_n \, dm_F = \lim_{n \to \infty} \sum_{j \in \mathbb{Z}} \phi_n(j).
$$

Since $\phi_n(j) \leq |f(j)|$, we have $\int |f| \, dm_F \leq \sum_{j \in \mathbb{Z}} |f(j)|$. On the other hand,

$$
\sum_{|j| \leq J} |f(j)| = \lim_{n \to \infty} \sum_{|j| \leq J} \phi_n(j) \leq \lim_{n \to \infty} \sum_{j \in \mathbb{Z}} \phi_n(j) = \int |f| \, dm_F,
$$

and so taking $J \to \infty$, $\sum_{j \in \mathbb{Z}} |f(j)| \leq \int |f| \, dm_F$. (Note: this last argument was really just the monotone convergence theorem again, for the case of counting measure on $\mathbb{Z}$.) So we conclude

$$
f \in L^1(m_F) \iff \sum_{j \in \mathbb{Z}} |f(j)| < \infty,$$

and then for such $f$ (by the same argument as for $|f|$, since we integrate complex functions by integrating $4$ positive ones),

$$
\int f \, dm_F = \sum_{j \in \mathbb{Z}} |f(j)|.
$$

3. Let $F : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and increasing, and let $m_F$ denote the corresponding Lebesgue-Stieltjes measure.

(a) Show that on any measure space $(X, \mathcal{M}, \mu)$, for any $g \in L^+$,

$$
\mu^{(g)} : \mathcal{M} \ni E \mapsto \int_E g \, d\mu \quad (:= \int \chi_E g \, d\mu)
$$

defines a measure on $(X, \mathcal{M})$.

Certainly $\mu^{(g)} : \mathcal{M} \to [0, \infty]$. Since $\chi_{\emptyset} = 0$, we have $\mu^{(g)}(\emptyset) = 0$. If $\{E_k\}_{k=1}^{\infty} \subset \mathcal{M}$ are disjoint, then $\chi_E g = \sum_{k=1}^{\infty} \chi_{E_k} g$. Then by countable additivity of the integral on $L^+$,

$$
\mu^{(g)}(E) = \int \sum_{k=1}^{\infty} \chi_{E_k} g = \sum_{k=1}^{\infty} \int \chi_{E_k} g = \sum_{k=1}^{\infty} \mu^{(g)}(E_k).
$$

(b) Show that on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$, $m_F = m^{(F')}$. Recall that Borel measures on $\mathbb{R}$ are uniquely determined by their (finite) values on bounded intervals $I = (a, b)$. Since usual Riemann integration of continuous functions on bounded intervals agrees with Lebesgue integration, by the Fundamental Theorem of Calculus,

$$
m_F(I) = F(b) - F(a) = \int_a^b F'(x) \, dx = \int_I F' = m^{(F')}(I),
$$

and so $m_F = m^{(F')}$. 

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4. Suppose $L^+ \supset \{f_n\}_{n=1}^\infty$ with $f_n \to f$ pointwise, and $\lim_{n \to \infty} \int f_n = \int f$.

(a) If $\int f < \infty$, show that for any $E \in \mathcal{M}$, $\lim_{n \to \infty} \int_E f_n = \int_E f$.

Since $0 \leq f_n \chi_E \to f \chi_E$ and $0 \leq f_n \chi_{E^c} \to f \chi_{E^c}$ pointwise, Fatou’s lemma yields
$$\int f \chi_E \leq \liminf \int f_n \chi_E, \quad \int f \chi_{E^c} \leq \liminf \int f_n \chi_{E^c}.$$ 

Since $\int f < \infty$, and so also $\int f_n < \infty$ (for $n$ large enough), we have
$$\int f \chi_E = \int f(1 - \chi_E) = \int f - \int f \chi_E, \quad \int f_n \chi_{E^c} = \int f_n - \int f_n \chi_E,$$ 

so
$$\liminf \int f_n \chi_E \geq \int f \chi_E = \int f - \liminf \int f_n \chi_E \geq \int f - \liminf \int f_n \chi_E = \int f - \liminf \left( \int f_n - \int f_n \chi_E \right) = \int f - \int f + \limsup \int f_n \chi_E = \limsup \int f_n \chi_E.$$ 

So the inequalities are equalities, and $\lim_{n \to \infty} \int_E f_n = \int_E f$.

(b) Show that the conclusion of part (a) may not hold if $\int f = \infty$.

Take Lebesgue measure on $\mathbb{R}$, $E = [0, 1]$, $f_n(x) = 1 + n \chi_{[0,1/n]}(x)$, and $f(x) = 1$. Then $f_n \to f$ pointwise, $\int f = \int f_n = \infty$ for all $n$, $\int_E f_n = 2$, while $\int_E f = 1$.

5. If $f \in L^1(\mu)$, show that for any $\epsilon > 0$, there is a $\delta > 0$ such that
$$\mu(E) < \delta \implies \int_E |f| d\mu < \epsilon.$$ 

If $f$ were bounded, $|f(x)| \leq M$, the conclusion would follow easily: $\int_E |f| \leq M \mu(E) < \epsilon$ if $\mu(E) < \delta := \frac{\epsilon}{M}$. For general $f \in L^1$, we approximate $|f|$ by bounded functions (there are several ways to do this): for $k = 1, 2, 3, \ldots$, set $f_k := |f| \chi_{\{|f(x)| \leq k\}} \leq k$. As $k \to \infty$, $|f| - f_k \to 0$ a.e. (since $f \in L^1 \implies \mu(\{|f| = \infty\}) = 0$), and $0 \leq |f| - f_k \leq |f| \in L^1$, so by the dominated convergence theorem, $\int (|f| - f_k) \to 0$.

Given $\epsilon > 0$, choose $k$ large enough so that $\int (|f| - f_k) < \frac{\epsilon}{2}$, and then choose $\delta = \frac{\epsilon}{2k}$. Then if $\mu(E) < \delta$,
$$\int_E |f| = \int_E (|f| - f_k + f_k) = \int_E (|f| - f_k) + \int_E f_k \leq \int (|f| - f_k) + k \mu(E) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$ 

6. Evaluate the following, with justification:

(a) For $\phi$ bounded and continuous, and $\psi \in L^1(\mu)$, $\lim_{n \to \infty} \int_{\mathbb{R}} \phi(x/n) \psi(x) \, dm(x)$

By continuity of $\phi$, the sequence of functions $\phi(x/n) \psi(x)$ converges a.e to $\phi(0) \psi(x)$. Since $\phi$ is bounded, $|\phi(x)| \leq M$, so $|\phi(x/n) \psi(x)| \leq M |\psi(x)| \in L^1(\mu)$. So by the Lebesgue dominated convergence theorem,
$$\lim_{n \to \infty} \int \phi(x/n) \psi(x) \, dm(x) = \int \phi(0) \psi(x) \, dm(x) = \phi(0) \int \psi(x) \, dm(x).$$
(b) For $\phi$ continuous and compactly supported, and $\psi \in L^1(m)$, \(\lim_{n \to \infty} \int_{-\infty}^{\infty} \phi(nx)\psi(x) \, dm(x)\)

Since $\phi$ is compactly supported, the sequence of functions $\phi(nx)\psi(x)$ converges pointwise to zero. Since $\phi$ is continuous and compactly supported, it is bounded, $|\phi(x)| \leq M$, so $|\phi(nx)\psi(x)| \leq M|\psi(x)| \in L^1(m)$. Lebesgue dominated convergence yields

$$\lim_{n \to \infty} \int \phi(nx)\psi(x) \, dm(x) = \int 0 \, dm(x) = 0.$$ 

(c) $\lim_{n \to \infty} \int_{-\infty}^{\infty} \frac{2}{x} \sin(x/n)e^{-|x|} \, dm(x)$

This is part (a) with $\phi(x) = \sin(x)/x$ (which is bounded and continuous – set $\phi(0) = 1$ to make it so), and $\psi(x) = e^{-|x|} \in L^1$, so

$$\lim_{n \to \infty} \int \frac{n}{x} \sin(x/n)e^{-|x|} \, dx = \int_{-\infty}^{\infty} e^{-|x|} \, dx = 2 \int_{0}^{\infty} e^{-x} \, dx = 2.$$ 

(d) $\lim_{n \to \infty} \int_{[0,1]} (1 + nx^2)(1 + x^2)^{-n} \, dm(x)$

Set $f_n(x) := (1 + nx^2)(1 + x^2)^{-n}$. We have

$$\lim_{n \to \infty} f_n(x)\chi_{[0,1]}(x) = \chi_{[0,1]} = 0 \text{ a.e.}.$$ 

Moreover $f_n(0) = 1$, and

$$f'_n(x) = -2n(n-1)x(1+x^2)^{-(n+1)} \leq 0 \quad x \in [0,1], \ n \geq 1,$$

so

$$|f_n\chi_{[0,1]}| \leq \chi_{[0,1]} \in L^1,$$

so Lebesgue dominated convergence says

$$\lim_{n \to \infty} \int_{0}^{1} (1 + nx^2)(1 + x^2)^{-n} \, dm(x) = \lim_{n \to \infty} \int f_n\chi_{[0,1]} \, dm = 0.$$ 

(Oct. 19)