Math 421/507: Assignment 2 Solutions

You may use any result from Chapters 0 ,1, or 2.1 of Folland, or established in class.

1. A more “classical” (than Lebesgue) way to measure “length” of sets $E \subset \mathbb{R}$, closely related to Riemann integration, is to consider coverings by finite (not countable) collections of intervals, leading to Jordan outer measure:

$$J^*(E) := \inf \left\{ \sum_{j=1}^{N} (b_j - a_j) \mid E \subset \bigcup_{j=1}^{N} (a_j, b_j), \ N \in \mathbb{N}, \ a_j \leq b_j \right\}$$

(a) Show $J^*$ is finitely subadditive, but not countably subadditive (hence not an outer measure in our sense).

For $E_1, E_2 \subset \mathbb{R}$, given $\epsilon > 0$, let $E_1 \subset \bigcup_{k=1}^{K} (a_k, b_k)$, $E_2 \subset \bigcup_{n=1}^{N} (c_n, d_n)$ such that $\sum_{k=1}^{K} (b_k - a_k) \leq J^*(E_1) + \epsilon$ and $\sum_{n=1}^{N} (d_n - c_n) \leq J^*(E_2) + \epsilon$. Then together $\{(a_k, b_k)\}$ and $\{(c_n, d_n)\}$ cover $E_1 \cup E_2$, so

$$J^*(E_1 \cup E_2) \leq \sum_{k=1}^{K} (b_k - a_k) + \sum_{n=1}^{N} (d_n - c_n) \leq J^*(E_1) + J^*(E_2) + 2\epsilon.$$

Since $\epsilon$ is arbitrary, $J^*(E_1 \cup E_2) \leq J^*(E_1) + J^*(E_2)$. This is finite subadditivity (as usual, the statement for the union of $n$ sets follows from the case $n = 2$ by induction).

Consider sets $\{j\}_{j \in \mathbb{Z}}$, so that $\bigcup_{j \in \mathbb{Z}} \{j\} = \mathbb{Z} \subset \mathbb{R}$. Since the single point $\{j\}$ can be covered by an arbitrarily short interval, $J^*(\{j\}) = 0$. However, no finite collection of finite intervals can cover $\mathbb{Z}$, so any finite cover of $\mathbb{Z}$ must contain an infinite interval, and so $J^*(\mathbb{Z}) = \infty$. Thus countable subadditivity fails.

(b) Show that $J^*(\mathbb{Q} \cap [0, 1]) = 1$ (here $\mathbb{Q}$ denotes the rationals).

Since $\epsilon > 0$, $\mathbb{Q} \cap [0, 1]$ is covered by the interval $(-\epsilon, 1 + \epsilon)$ of length $1 + 2\epsilon$, we have $J^*(\mathbb{Q} \cap [0, 1]) \leq 1 + 2\epsilon$, and hence $J^*(\mathbb{Q} \cap [0, 1]) \leq 1$.

Now fix $\epsilon > 0$, and suppose $\mathbb{Q} \cap [0, 1] \subset U = \bigcup_{j=1}^{N} (a_j, b_j)$. Then $[0, 1] \subset U_{\epsilon} := \bigcup_{j=1}^{N} (a_j - \frac{\epsilon}{N}, b_j + \frac{\epsilon}{N})$, for any $x \in [0, 1] \setminus U_{\epsilon}$, then the interval $(x - \frac{\epsilon}{N}, x + \frac{\epsilon}{N})$, which must contain a point of $\mathbb{Q} \cap [0, 1]$, is contained in $U^{c}$, a contradiction. So

$$1 = J^*([0, 1]) \leq \sum_{j=1}^{N} (b_j - a_j + 2\epsilon/N) = \sum_{j=1}^{N} (b_j - a_j) + 2\epsilon.$$

Thus $J^*(\mathbb{Q} \cap [0, 1]) \geq 1 - 2\epsilon$. Hence $J^*(\mathbb{Q} \cap [0, 1]) \geq 1$.

(c) A set $E \subset \mathbb{R}$ is said to be Jordan measurable if its Jordan outer measure $J^*(E)$ agrees with its Jordan inner measure

$$J_*(E) := \sup \left\{ \sum_{j=1}^{N} (b_j - a_j) \mid E \supset \bigcup_{j=1}^{N} (a_j, b_j), \ N \in \mathbb{N}, \ a_j \leq b_j \right\}. $$

Show that $\mathbb{Q} \cap [0, 1]$ is not Jordan measurable.

Since $\mathbb{Q} \cap [0, 1]$ contains no non-empty open interval, $J_*(\mathbb{Q} \cap [0, 1]) = 0$. Combined with the previous part, this shows $\mathbb{Q} \cap [0, 1]$ is not Jordan measurable.
Recalling that \( \mathbb{Q} \cap [0,1] \) (indeed any countable set) is Lebesgue measurable (in fact Borel) with \( m(\mathbb{Q} \cap [0,1]) = 0 \), this exercise shows an advantage in allowing the countable covers of Lebesgue measure, over the finite covers of classical Jordan “measure”.

2. (a) Construct an open set of arbitrarily small measure containing \( \mathbb{Q} \cap [0,1] \).

Let \( \{q_1, q_2, q_3, \ldots\} \) be an enumeration of the (countable) set \( \mathbb{Q} \cap [0,1] \). For any \( \epsilon > 0 \),

\[
U_\epsilon := \bigcup_{k=1}^{\infty} (q_k - \frac{\epsilon}{2^k}, q_k + \frac{\epsilon}{2^k})
\]

is an open set, containing \( \mathbb{Q} \cap [0,1] \), with

\[
m(U_\epsilon) \leq \sum_{k=1}^{\infty} \frac{2\epsilon}{2^k} = 2\epsilon.
\]

(b) Construct a nowhere dense (closure contains no open set) subset of \( [0,1] \) of measure arbitrarily close to 1.

With \( U_\epsilon \) as above, \( [0,1] \setminus U_\epsilon \) is closed, and since it contains no rational, it contains no open interval, so no open set, and hence is nowhere dense. Moreover,

\[
m([0,1] \setminus U_\epsilon) \geq m([0,1]) - m(U_\epsilon) \geq 1 - 2\epsilon.
\]

(c) Show there cannot exist a nowhere dense set \( E \subset [0,1] \) with \( m(E) = 1 \).

If so, \( \tilde{E} \subset [0,1] \) contains no open set, and \( m(\tilde{E}) \geq m(E) = 1 \), hence \( m(\tilde{E}) = 1 \). Then \( U = (0,1) \cap (\tilde{E})^c \) is open, and non-empty (if empty, then \( (0,1) \subset \tilde{E} \), but \( \tilde{E} \) contains no open set), so must contain an open interval, and so \( m(U) > 0 \). But this contradicts \( 1 = m((0,1)) = m(U) + m((0,1) \cap \tilde{E}) = m(U) + 1 \).

(d) If we were to try to define a “Lebesgue inner measure” by

\[
m_*(E) := \sup \left\{ \sum_{j=1}^{\infty} (b_j - a_j) \mid E \supset \bigcup_{j=1}^{\infty} (a_j, b_j), \ a_j \leq b_j \right\},
\]

compute it for your set from (b) (and so conclude this definition is not good).

A nowhere dense set contains no non-empty open interval, so its \( m_* \) (by this definition) would be 0 (and so in disagreement with its actual Lebesgue measure).

(e) Use the fact (proved in class) that given a measurable set \( E \subset \mathbb{R} \), and \( \epsilon > 0 \), there is a closed set \( F \subset E \) with \( m(E \setminus F) \leq \epsilon \), to show that

\[
m(E) = \sup \{ m(K) \mid E \supset K \text{ compact} \}.
\]

So this would be a better notion of “inner measure”.

Set \( S := \sup \{ m(K) \mid E \supset K \text{ compact} \} \). For any compact \( K \subset E \), \( m(K) \leq m(E) \), hence \( S \leq m(E) \).
3. (a) If $\mu$ is a Borel measure on $\mathbb{R}$ which is finite on bounded sets, show the function

$$F(x) := \begin{cases} 
\mu((0,x]) & x > 0 \\
0 & x = 0 \\
-\mu((x,0]) & x < 0
\end{cases}$$

is increasing ($x < y \implies F(x) \leq F(y)$) and right continuous ($F(x+) = F(x)$).

Considering all cases, we see that if $x < y$, then $F(y) - F(x) = \mu((x,y]) \geq 0$, so $F$ is increasing.

Fix $x_0$. If $x_0 \leq x_j \to x_0$, then since $\cap_{j=1}^{\infty}(x_0, x_j] = \emptyset$,

$$F(x_j) - F(x_0) = \mu((x_0, x_j]) = \mu(\emptyset) = 0,$$

by continuity from above of measures. So $F$ is right continuous.

(b) For an increasing, right continuous function $F$ on $\mathbb{R}$, let $m_F$ denote the corresponding Lebesgue-Stieltjes measure on $\mathbb{R}$.

i. Under what conditions on $F$ is $m_F$ finite?

$m_F$ is finite iff $\infty > m_F(\mathbb{R}) = F(\infty) - F(-\infty)$, i.e. iff $F$ is bounded.

ii. Let $x_0 \in \mathbb{R}$. Under what conditions on $F$ is $m_F(\{x_0\}) = 0$?

$m_F(\{x_0\}) = 0$ iff $F$ is continuous at $x_0$.

To see this, note that for any $x_0 > x_j \to x_0$, $\{x_0\} = \cap_{j=1}^{\infty}(x_j, x_0]$, and so by continuity from above of $m_F$,

$$m_F(\{x_0\}) = \lim_{j \to \infty} m_F((x_j, x_0]) = \lim_{j \to \infty} (F(x_0) - F(x_j)) = F(x_0) - \lim_{j \to \infty} F(x_j).$$

Thus $F$ is left continuous at $x_0$ iff $m_F(\{x_0\}) = 0$, and the statement follows.

iii. Identify $m_F$ when $F(x) = |x|$.

We have $m_F = \sum_{j \in \mathbb{Z}} \delta_j$. To prove this, it suffices to check they agree on $h$-intervals $(a, b]$, $a < b$ (since the measure of $h$-intervals uniquely determines a Borel measure on $\mathbb{R}$). Well,

$$\sum_{j \in \mathbb{Z}} \delta_j((a, b]) = \# \text{ of integers in } (a, b] = |b| - |a|. $$
4. For $d \in [0, 1]$, define the $d$-dimensional Hausdorff outer measure $H_d(E)$ of a set $E \subset \mathbb{R}$:

$$H_{d,\delta}(E) := \inf \left\{ \sum_{j=1}^{\infty} l_j^d \mid E \subset \bigcup_{j=1}^{\infty} (a_j, a_j + l_j), \ 0 \leq l_j \leq \delta \right\}, \ H_d(E) := \sup_{\delta > 0} H_{d,\delta}(E).$$

(a) Show that $H_d$ is an outer measure.

Clearly $H_{d,\delta}(\emptyset) = 0$ for all $\delta$, hence $H_d(\emptyset) = 0$.

For the monotonicity: if $E \subset F$, $H_{d,\delta}(E) \leq H_{d,\delta}(F)$ (since the inf for $E$ is taken over a larger set), so $H_{d,\delta}(E) \leq H_d(F)$. Then taking $\sup_{\delta}$, $H_d(E) \leq H_d(F)$.

For subadditivity: let $\{E_k\}_{k=1}^{\infty}$ be subsets. Fix $\delta > 0$. Fix $\epsilon > 0$. Let $E_k \subset \bigcup_{j=1}^{\infty} (a_k^j, a_k^j + l_k^j)$ with $l_k^j \leq \delta$ and $\sum_{j=1}^{\infty} (l_k^j)^d \leq H_{d,\delta}(E_k) + \epsilon 2^{-k}$. Then $\cup_k E_k \subset \bigcup_{k,j} (a_k^j, a_k^j + l_k^j)$, and so

$$H_{d,\delta}(\cup_k E_k) \leq \sum_{k,j} (l_k^j)^d \leq \sum_k \left( H_{d,\delta}(E_k) + \epsilon 2^{-k} \right) = \sum_k H_{d,\delta}(E_k) + \epsilon.$$

Since $\epsilon$ was arbitrary, $H_{d,\delta}(\cup_k E_k) \leq \sum_k H_{d,\delta}(E_k)$ (this shows $H_{d,\delta}$ is an outer measure).

Now since $H_{d,\delta}(E_k) \leq H_d(E_k)$, we have $H_{d,\delta}(\cup_k E_k) \leq \sum_k H_d(E_k)$ for each $\delta > 0$. Then taking $\sup_{\delta}$, we arrive at $H_d(\cup_k E_k) \leq \sum_k H_d(E_k)$ as required.

(b) Identify $H_0$.

$H_0$ is just counting measure. To see this: if $E = \{x_1, x_2, \ldots, x_N\}$ is finite, then for any $\delta > 0$, $E \subset \bigcup_{j=1}^{N} (x_j - \frac{\delta}{2}, x_j + \frac{\delta}{2})$, so $H_{d,\delta}(E) \leq N$. Thus $H_d(E) \leq N$.

Conversely, if $\delta < \min_{j \neq k} |x_j - x_k|$, then any covering of $E$ by intervals of length $\leq \delta$ must contain at least $N$ intervals. So $H_{0,\delta}(E) \geq N$. So $H_0(E) \geq N$.

Thus $H_0(E) = N$. Moreover, the previous argument shows that any set containing (at least) $N$ points has $H_0 \geq N$, which shows that $H_0(E) = \infty$ when $E$ is infinite.

(c) Show that for the Cantor 1/3-set $C$, $H_d(C) = 0$ if $d > \frac{\log 2}{\log 3}$.

Let $\delta > 0$ and let $0 < \epsilon < 1$. Let $K_j$, $j = 0, 1, 2, 3, \ldots$ denote the (disjoint) union of $2^j$ closed intervals of length $3^{-j}$ in the $j$-th stage of the Cantor set construction, so that $C = \cap_{j=0}^{\infty} K_j$. Let $U_j$ be the union of open intervals of length $3^{-j}(1 + \epsilon)$ centred at the centres of the intervals making up $K_j$. Then $C \subset K_j \subset U_j$, and so for any $j$ large enough that $2 \cdot 3^{-j} \leq \delta$, we have

$$H_{d,\delta}(C) \leq 2^j \left(3^{-j}(1 + \epsilon)\right)^d = \left(\frac{2}{3^d}\right)^j (1 + \epsilon)^d.$$

Since $\epsilon$ was arbitrary, $H_{d,\delta}(C) \leq \left(\frac{2}{3^d}\right)^j \to 0$ as $j \to \infty$, provided $2 < 3^d$, i.e. $d > \frac{\log 2}{\log 3}$. So in this case, $H_{d,\delta}(C) = 0$ for all $\delta$, and so $H_d(C) = 0$.

5. Fix a sequence $\vec{\alpha} := \{\alpha_j\}_{j=1}^{\infty} \subset (0, 1)$, and consider the generalized Cantor set $C_{\vec{\alpha}}$ constructed by starting with $K_0 := [0, 1]$, removing the “open middle $\alpha_1$-th” of $K_0$ to produce $K_1$, removing the “open middle $\alpha_2$-th” of both intervals of $K_1$ to produce $K_2$, etc., and then setting $C_{\vec{\alpha}} := \cap_{j=1}^{\infty} K_j$ (so the standard Cantor 1/3-set corresponds to $\vec{\alpha} = \{1/3, 1/3, 1/3, 1/3, \ldots\}$).
(a) Compute \( m(C_{\vec{\alpha}}) \).

Removing the middle \( \alpha_j \)-th of each interval at the \( j \)-th stage reduces the measure by a factor of \( 1 - \alpha_j \), so:

\[
m(C_{\vec{\alpha}}) = (1 - \alpha_1)(1 - \alpha_2)(1 - \alpha_3) \cdots = \prod_{j=1}^{\infty} (1 - \alpha_j).
\]

(b) Show that for any \( \beta \in [0, 1) \), some choice of \( \vec{\alpha} \) results in \( m(C_{\vec{\alpha}}) = \beta \).

If \( \beta = 0 \), taking a constant sequence \( \alpha_1 = \alpha_2 = \alpha_3 = \cdots \in (0, 1) \) does it (for example our Cantor 1/3 set). For \( \beta \in (0, 1) \), take \( \alpha_j = 1 - \beta(2^{-j}) \in (0, 1) \) so that

\[
m(C_{\vec{\alpha}}) = \prod_{j=1}^{\infty} (1 - \alpha_j) = \exp \left( \sum_{j=1}^{\infty} \log(1 - \alpha_j) \right) = \exp \left( \log(\beta) \sum_{j=1}^{\infty} 2^{-j} \right) = e^{\log(\beta)} = \beta.
\]

(c) Show that \( C_{\vec{\alpha}} \) is nowhere dense (its closure contains no open set).

Being an intersection of closed sets, \( C_{\vec{\alpha}} \) is closed, so it suffices to show it contains no open set. If it does, it contains an open interval \((a, b) \subset C_{\vec{\alpha}}\). The set \( K_j \) is a disjoint union of closed intervals of length \( 2^{-j} \prod_{k=1}^{j} (1 - \alpha_k) \to 0 \) as \( j \to \infty \), so for \( j \) large enough, \((a, b) \not\subset K_j\), and so \((a, b) \not\subset C_{\vec{\alpha}}\).

6. Show that a decreasing (or increasing) function \( f : \mathbb{R} \to \mathbb{R} \) is Borel measurable.

If \( f \) is increasing, then for any \( a \in \mathbb{R} \), \( f^{-1}((a, \infty)) \) is of the form \((b, \infty)\) or \([b, \infty)\) for some \( b \in \mathbb{R} \) (or \( b = -\infty \)), in any case a Borel set. Since sets of the form \((a, \infty)\) generate the Borel sets, \( f \) is Borel measurable. A similar argument covers the decreasing case.

7. Let \((X, \mathcal{M})\) be a measurable space. Show that if \( f : X \to \mathbb{R} \) is such that \( f^{-1}((r, \infty)) \in \mathcal{M} \) for each \( r \in \mathbb{Q} \), then \( f \) is measurable.

For any \( a \in \mathbb{R} \), let \( \mathbb{Q} \ni r_j \to a \) as \( j \to \infty \) with \( r_j \leq a \) (by density of \( \mathbb{Q} \)). Then

\[
f^{-1}([a, \infty]) = \bigcap_{j=1}^{\infty} f^{-1}((r_j, \infty])
\]

and so is measurable. Thus \( f \) is measurable.

8. Consult [Folland, p 20] for the definition of a set \( N \subset [0, 1) \) (whose construction rests on the axiom of choice) such that no two points in \( N \) differ by a rational, and \([0, 1) = \bigcup_{q \in \mathbb{Q} \cap [0,1)} N_q\), where

\[
N_q := \left\{ \begin{array}{ll}
x + q & \text{if } x + q < 1 \\
x + q - 1 & \text{if } x + q \geq 1
\end{array} \right\} \quad |x \in N\}
\]

denotes the set \( N + q \) (mod 1). Show that:
(a) \( N \) is not measurable: \( N \not\in \mathcal{L} \); 

If \( N \) were measurable, then since the the \( N_q \) (for \( q \in \mathbb{Q} \cap [0, 1) \)) are disjoint, and by translation invariance of Lebesgue measure, 

\[
1 = m(\mathbb{R}) = \sum_{q \in \mathbb{Q} \cap [0, 1)} m(N_q) = \sum_{q \in \mathbb{Q} \cap [0, 1)} m(N)
\]

which would contradict both \( m(N) = 0 \) and \( m(N) > 0 \).

(b) \( 0 < m^*(N) \leq 1 \); 

By monotonicity of outer measure, \( N \subset [0, 1) \) implies \( m^*(N) \leq 1 \), and by subadditivity and translation invariance of \( m^* \), 

\[
1 = m^*(\mathbb{R}) \leq \sum_{q \in \mathbb{Q} \cap [0, 1)} m^*(N_q) = \sum_{q \in \mathbb{Q} \cap [0, 1)} m^*(N),
\]

which implies \( m^*(N) > 0 \).

(c) if \( \mathcal{L} \ni E \subset N \), then \( m(E) = 0 \); 

Let \( E_q \subset [0, 1) \) denote the translate \( E + q \mod 1 \) as defined above for \( N \). Since \( E \subset N \), the sets \( \{E_q\}_{q \in \mathbb{Q} \cap [0, 1)} \) are disjoint, and by translation invariance they all have the same measure, so 

\[
1 \geq m(\bigcup_{q \in \mathbb{Q} \cap [0, 1)} E_q) = \sum_{q \in \mathbb{Q} \cap [0, 1)} m(E_q) = \sum_{q \in \mathbb{Q} \cap [0, 1)} m(E)
\]

which forces \( m(E) = 0 \).

(d) if \( E \in \mathcal{L} \) with \( m(E) > 0 \), then \( E \) has a non-measurable subset; 

Since \( \bigcup_{q \in \mathbb{Q} \cap [0, 1)} N_q = [0, 1) \), we have \( E = \bigcup_{q \in \mathbb{Q} \cap [0, 1)} E \cap N_q \) (disjoint union). If all the sets \( E \cap N_q \) are measurable, then 

\[
0 < m(E) = m(\bigcup_{q \in \mathbb{Q} \cap [0, 1)} E \cap N_q) = \sum_{q \in \mathbb{Q} \cap [0, 1)} m(E \cap N_q),
\]

and so \( m(E \cap N_q) > 0 \) for some \( q \). But then \( E_q \cap N \) is a measurable subset of \( N \) with positive measure, contradicting part (c). Thus at least one of the \( E \cap N_q \) is a non-measurable subset of \( E \).

(e) \( m^*([0, 1) \setminus N) = 1 \); 

If \( B \) is a union of \( h \)-intervals covering \([0, 1) \setminus N \), then \( B^c \cap [0, 1) \subset N \). Since \( B^c \cap [0, 1) \) is Borel, hence measurable, by (c) we have \( m(B^c \cap [0, 1) \) = 0. Then since 

\[
1 = m([0, 1)) = m(B \cap [0, 1)) + m(B^c \cap [0, 1)), \text{ we have } m(B) \geq m(B \cap [0, 1)) = 1.
\]

Hence \( m^*([0, 1) \setminus N) = 1 \). Remark: this shows that the "inner measure" of \( N \) is 0.

(Oct. 4)