Math 421/507: Assignment 2 (Due Friday, Oct.5)

You may use any result from Chapters 0 ,1, or 2.1 of Folland, or established in class.

1. A more "classical" (than Lebesgue) way to measure "length" of sets $E \subset \mathbb{R}$, closely related to Riemann integration, is to consider converings by *finite* (not countable) collections of intervals, leading to *Jordan outer measure*:

$$J^*(E) := \inf \left\{ \sum_{j=1}^{N} (b_j - a_j) \mid E \subset \bigcup_{j=1}^{N} (a_j, b_j), \quad N \in \mathbb{N}, \quad a_j \leq b_j \right\}$$

   (a) Show J^* is finitely subadditive, but *not* countably subadditive (hence not an *outer measure* in our sense).

   (b) Show that $J^*(\mathbb{Q} \cap [0,1]) = 1$ (here $\mathbb{Q}$ denotes the rationals).

   (c) A set $E \subset \mathbb{R}$ is said to be *Jordan measurable* if its Jordan outer measure $J^*(E)$ agrees with its *Jordan inner measure*

$$J_*(E) := \sup \left\{ \sum_{j=1}^{N} (b_j - a_j) \mid E \supset \bigcup_{j=1}^{N} (a_j, b_j), \quad N \in \mathbb{N}, \quad a_j \leq b_j \right\}.$$

   Show that $\mathbb{Q} \cap [0,1]$ is not Jordan measurable.

   Recalling that $\mathbb{Q} \cap [0,1]$ (indeed any countable set) is Lebesgue measurable (in fact Borel) with $m(\mathbb{Q} \cap [0,1]) = 0$, this exercise shows an advantage in allowing the countable covers of Lebesgue measure, over the finite covers of classical Jordan "measure".

2. (a) Construct an open set of arbitrarily small measure containing $\mathbb{Q} \cap [0,1]$.

   (b) Construct a nowhere dense (closure contains no open set) subset of $[0,1]$ of measure arbitrarily close to 1.

   (c) Show there cannot exist a nowhere dense set $E \subset [0,1]$ with $m(E) = 1$.

   (d) If we were to try to define a "Lebesgue inner measure" by

$$m_*(E) := \sup \left\{ \sum_{j=1}^{\infty} (b_j - a_j) \mid E \supset \bigcup_{j=1}^{\infty} (a_j, b_j), \quad a_j \leq b_j \right\},$$

   compute it for your set from (b) (and so conclude this definition is not good).

   (e) Use the fact (proved in class) that given a measurable set $E \subset \mathbb{R}$, and $\epsilon > 0$, there is a closed set $F \subset E$ with $m(E \backslash F) \leq \epsilon$, to show that

$$m(E) = \sup\{m(K) \mid E \supset K \text{ compact } \}.$$

   So this would be a better notion of "inner measure".

3. (a) If μ is a Borel measure on $\mathbb{R}$ which is finite on bounded sets, show the function

$$F(x) := \begin{cases} \mu((0,x]) & x > 0 \\ 0 & x = 0 \\ -\mu((x,0]) & x < 0 \end{cases}$$

   is increasing ($x < y \implies f(x) \leq f(y)$) and right continuous ($f(x+) = f(x)$).

(b) For an increasing, right continuous function F on $\mathbb{R}$, let m_F denote the corresponding Lebesgue-Stieltjes measure on $\mathbb{R}$.

i. Under what conditions on F is m_F finite?

ii. Let $x_0 \in \mathbb{R}$. Under what conditions on F is $m_F(\{x_0\}) = 0$?

iii. Identify m_F when $F(x) = \lfloor x \rfloor$.

4. For $d \in [0,1]$, define the *d-dimensional Hausdorff outer measure* $H_d(E)$ of a set $E \subset \mathbb{R}$:

$$H_{d,\delta}(E) := \inf \left\{ \sum_{j=1}^{\infty} l_j^d \mid E \subset \bigcup_{j=1}^{\infty} (a_j, a_j + l_j),\ 0 \le l_j \le \delta \right\}, \quad H_d(E) := \sup_{\delta > 0} H_{d,\delta}(E).$$

(a) Show that H_d is an outer measure.

(b) Identify H_0.

(c) Show that for the Cantor 1/3-set C, $H_d(C) = 0$ if $d > \frac{\log 2}{\log 3}$.

5. Fix a sequence $\vec{\alpha} := \{\alpha_j\}_{j=1}^{\infty} \subset (0,1)$, and consider the generalized Cantor set $C_{\vec{\alpha}}$ constructed by starting with $K_0 := [0,1]$, removing the "open middle α_1-th" of K_0 to produce K_1, removing the "open middle α_2-th" of both intervals of K_1 to produce K_2, etc., and then setting $C_{\vec{\alpha}} := \cap_{j=1}^{\infty} K_j$ (so the standard Cantor 1/3-set corresponds to $\vec{\alpha} = \{1/3, 1/3, 1/3, 1/3, \ldots\}$).

(a) Compute $m(C_{\vec{\alpha}})$.

(b) Show that for any $\beta \in [0,1)$, some choice of $\vec{\alpha}$ results in $m(C_{\vec{\alpha}}) = \beta$.

(c) Show that $C_{\vec{\alpha}}$ is nowhere dense (its closure contains no open set).

6. Show that a decreasing (or increasing) function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable.

7. Let $(X, \mathcal{M})$ be a measurable space. Show that if $f : X \to \bar{\mathbb{R}}$ is such that $f^{-1}((r, \infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$, then f is measurable.

8. Consult [Folland, p 20] for the definition of a set $N \subset [0,1)$ (whose construction rests on the axiom of choice) such that no two points in N differ by a rational, and $[0,1) = \cup_{q \in \mathbb{Q} \cap [0,1)} N_q$, where

$$N_q := \left\{ \left\{ \begin{array}{ll} x + q & \text{if } x + q < 1 \\ x + q - 1 & \text{if } x + q \ge 1 \end{array} \right. \mid x \in N \right\}$$

denotes the set $N + q \pmod 1$. Show that:

(a) N is not measurable: $N \notin \mathcal{L}$;

(b) $0 < m^*(N) \le 1$;

(c) if $\mathcal{L} \ni E \subset N$, then $m(E) = 0$;

(d) if $E \in \mathcal{L}$ with $m(E) > 0$, then E has a non-measurable subset;

(e) $m^*([0,1) \backslash N) = 1$;

(Sep. 24)