Math 421/507: Assignment 2 (Due Friday, Oct.5)

You may use any result from Chapters 0, 1, or 2.1 of Folland, or established in class.

1. A more “classical” (than Lebesgue) way to measure “length” of sets $E \subset \mathbb{R}$, closely related to Riemann integration, is to consider coverings by finite (not countable) collections of intervals, leading to Jordan outer measure:

$$J^*(E) := \inf \left\{ \sum_{j=1}^{N} (b_j - a_j) \mid E \subset \bigcup_{j=1}^{N} (a_j, b_j), \ N \in \mathbb{N}, \ a_j \leq b_j \right\}$$

(a) Show $J^*$ is finitely subadditive, but not countably subadditive (hence not an outer measure in our sense).

(b) Show that $J^*(\mathbb{Q} \cap [0,1]) = 1$ (here $\mathbb{Q}$ denotes the rationals).

(c) A set $E \subset \mathbb{R}$ is said to be Jordan measurable if its Jordan outer measure $J^*(E)$ agrees with its Jordan inner measure

$$J_*(E) := \sup \left\{ \sum_{j=1}^{N} (b_j - a_j) \mid E \supset \bigcup_{j=1}^{N} (a_j, b_j), \ N \in \mathbb{N}, \ a_j \leq b_j \right\} .$$

Show that $\mathbb{Q} \cap [0,1]$ is not Jordan measurable.

Recalling that $\mathbb{Q} \cap [0,1]$ (indeed any countable set) is Lebesgue measurable (in fact Borel) with $m(\mathbb{Q} \cap [0,1]) = 0$, this exercise shows an advantage in allowing the countable covers of Lebesgue measure, over the finite covers of classical Jordan “measure”.

2. (a) Construct an open set of arbitrarily small measure containing $\mathbb{Q} \cap [0,1]$.

(b) Construct a nowhere dense (closure contains no open set) subset of $[0,1]$ of measure arbitrarily close to 1.

(c) Show there cannot exist a nowhere dense set $E \subset [0,1]$ with $m(E) = 1$.

(d) If we were to try to define a “Lebesgue inner measure” by

$$m_*(E) := \sup \left\{ \sum_{j=1}^{\infty} (b_j - a_j) \mid E \supset \bigcup_{j=1}^{\infty} (a_j, b_j), \ a_j \leq b_j \right\} ,$$

compute it for your set from (b) (and so conclude this definition is not good).

(e) Use the fact (proved in class) that given a measurable set $E \subset \mathbb{R}$, and $\epsilon > 0$, there is a closed set $F \subset E$ with $m(E \setminus F) \leq \epsilon$, to show that

$$m(E) = \sup \{ m(K) \mid E \supset K \text{ compact} \} .$$

So this would be a better notion of “inner measure”.

3. (a) If $\mu$ is a Borel measure on $\mathbb{R}$ which is finite on bounded sets, show the function

$$F(x) := \begin{cases} \mu((0,x]) & x > 0 \\ 0 & x = 0 \\ -\mu((x,0]) & x < 0 \end{cases}$$

is increasing ($x < y \implies f(x) \leq f(y)$) and right continuous ($f(x+) = f(x)$).
(b) For an increasing, right continuous function $F$ on $\mathbb{R}$, let $m_F$ denote the corresponding Lebesgue-Stieltjes measure on $\mathbb{R}$.

i. Under what conditions on $F$ is $m_F$ finite?

ii. Let $x_0 \in \mathbb{R}$. Under what conditions on $F$ is $m_F(\{x_0\}) = 0$?

iii. Identify $m_F$ when $F(x) = \lfloor x \rfloor$.

4. For $d \in [0, 1]$, define the $d$-dimensional Hausdorff outer measure $H_d(E)$ of a set $E \subset \mathbb{R}$:

$$H_{d,\delta}(E) := \inf \left\{ \sum_{j=1}^{\infty} l_j^d : E \subseteq \bigcup_{j=1}^{\infty} (a_j, a_j + l_j), 0 \leq l_j \leq \delta \right\}, \quad H_d(E) := \sup_{\delta>0} H_{d,\delta}(E).$$

(a) Show that $H_d$ is an outer measure.

(b) Identify $H_0$.

(c) Show that for the Cantor $1/3$-set $C$, $H_d(C) = 0$ if $d > \frac{\log 2}{\log 3}$.

5. Fix a sequence $\vec{\alpha} := \{\alpha_j\}_{j=1}^{\infty} \subset (0, 1)$, and consider the generalized Cantor set $C_{\vec{\alpha}}$ constructed by starting with $K_0 := [0, 1]$, removing the “open middle $\alpha_1$-th” of $K_0$ to produce $K_1$, removing the “open middle $\alpha_2$-th” of both intervals of $K_1$ to produce $K_2$, etc., and then setting $C_{\vec{\alpha}} := \cap_{j=1}^{\infty} K_j$ (so the standard Cantor $1/3$-set corresponds to $\vec{\alpha} = \{1/3, 1/3, 1/3, 1/3, \ldots\}$).

(a) Compute $m(C_{\vec{\alpha}})$.

(b) Show that for any $\beta \in [0, 1)$, some choice of $\vec{\alpha}$ results in $m(C_{\vec{\alpha}}) = \beta$.

(c) Show that $C_{\vec{\alpha}}$ is nowhere dense (its closure contains no open set).

6. Show that a decreasing (or increasing) function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable.

7. Let $(X, \mathcal{M})$ be a measurable space. Show that if $f : X \to \bar{\mathbb{R}}$ is such that $f^{-1}((r, \infty)) \in \mathcal{M}$ for each $r \in \mathbb{Q}$, then $f$ is measurable.

8. Consult [Folland, p 20] for the definition of a set $N \subset [0, 1)$ (whose construction rests on the axiom of choice) such that no two points in $N$ differ by a rational, and $[0, 1) = \cup_{q \in \mathbb{Q} \cap [0,1)} N_q$, where

$$N_q := \left\{ \begin{array}{ll} x + q & \text{if } x + q < 1 \\ x + q - 1 & \text{if } x + q \geq 1 \end{array} \right| x \in N \right\}$$

denotes the set $N + q$ (mod 1). Show that:

(a) $N$ is not measurable: $N \notin \mathcal{L}$;
(b) $0 < m^*(N) \leq 1$;
(c) if $\mathcal{L} \ni E \subset N$, then $m(E) = 0$;
(d) if $E \in \mathcal{L}$ with $m(E) > 0$, then $E$ has a non-measurable subset;
(e) $m^*([0,1)\setminus N) = 1$;

(Sep. 24)