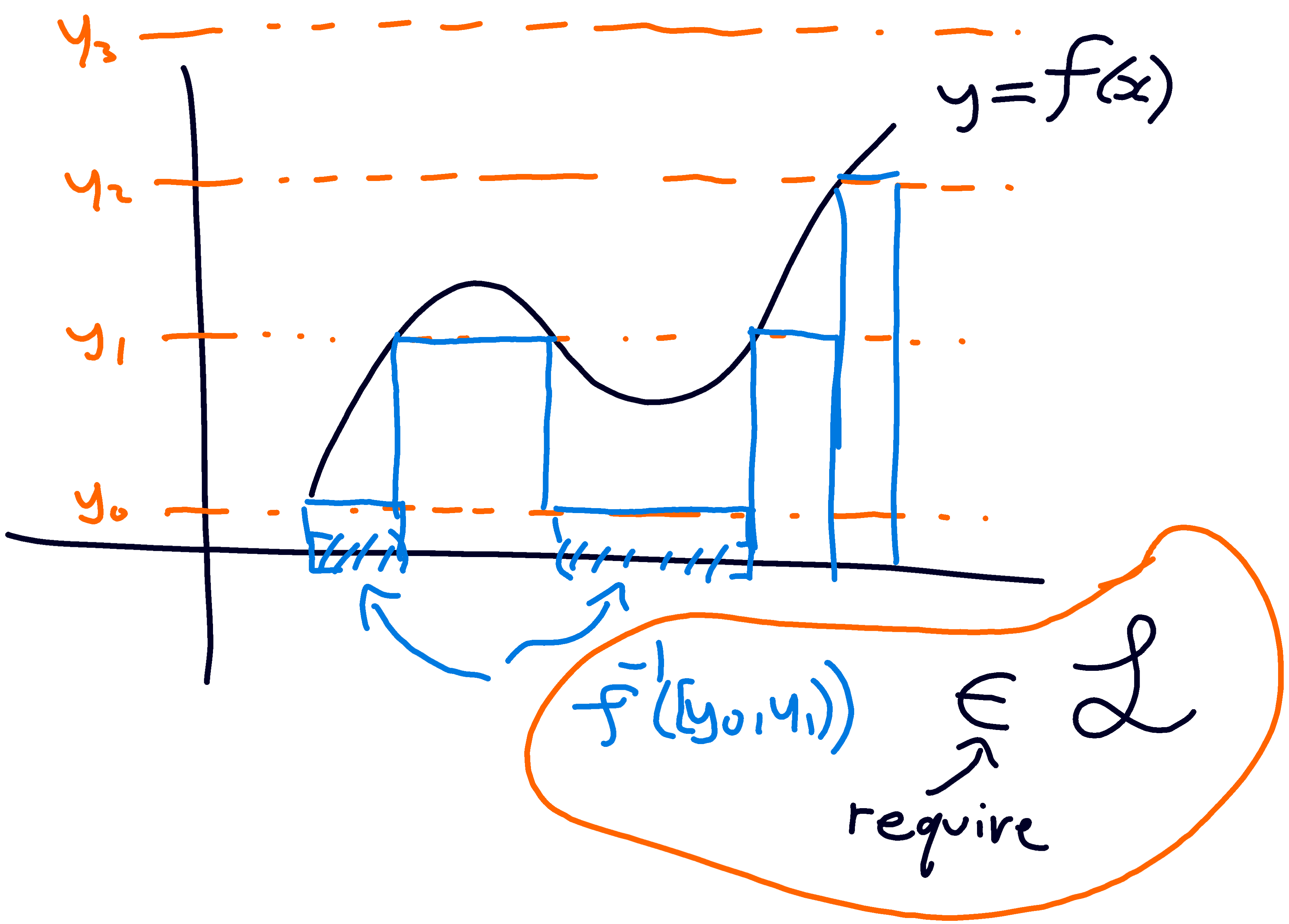
$\boxed{B}$ Integration (2.1–2.6)



Measurable Functions (2.1)

Def: 1. $(X, \mathcal{M})$, $(Y, \mathcal{N})$ measurable spaces

set, σ-alg

$f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$ measurable if

$$f^{-1}(E) \in \mathcal{M} \ \forall \ E \in \mathcal{N}$$

2. $f : X \to \mathbb{R}$ (or $\mathbb{C}$) is measurable if it is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ (or $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$) measurable

3. $f: \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$) is <u>Borel measurable</u>

if it is $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}})$-meas (or $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{C}})$-meas)

4. $f: \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$) is <u>Lebesgue measurable</u>

if it is $(\mathcal{L}, \mathcal{B}_{\mathbb{R}})$ (or $(\mathcal{L}, \mathcal{B}_{\mathbb{C}})$)-meas.

<u>Rems:</u> 1. if $\mathcal{N}$ is generated by $\mathcal{E} \subset \mathcal{N}$,

$f: X \to Y$ is measurable $\iff$

$f^{-1}(E) \in \mathcal{M} \ \forall E \in \mathcal{E}$ (exercise) in part., $f: X \to \mathbb{R}$ is meas.

$\Longleftrightarrow f^{-1}((a,\infty)) \in \mathcal{M} \quad \forall a$

$\Longleftrightarrow f^{-1}([a,\infty))$ " " " "

$\Longleftrightarrow f^{-1}((-\infty,a))$ " " " "

$\Longleftrightarrow f^{-1}((-\infty,a])$ " " " "

2. $f: \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$) continuous

$\Rightarrow$ Borel measurable

3. composition of meas. is meas: $(X,\mathcal{M}), (Y,\mathcal{N}), (Z,\Theta)$

$f: X \to Y$, $g: Y \to Z$ meas

$\Rightarrow$ $g \circ f : X \to Z$ is meas (exercise)

ex: • $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$ Borel meas $\Rightarrow$ $g \circ f$ Borel meas.

• " " Lebesgue " $\not\Rightarrow$ $g \circ f$ Lebesgue meas

Detour: product σ-alg's: $\cdot\ \{(Y_\alpha, \mathcal{M}_\alpha)\}_{\alpha\in A}$ meas. spaces

$\cdot\ X$, a set

$\cdot\ f_\alpha : X \to Y_\alpha$

Def: • σ-alg. gen. by $\{f_\alpha\}$ is the " " " sets $f_\alpha^{-1}(E)$, $E \in \mathcal{M}_\alpha$

• if $X = \prod_{\alpha\in A} Y_\alpha$, the product σ-alg $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha$ is gen. by coordinate maps $\pi_\alpha : X \to Y_\alpha$

(if A is countable, $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha$ is gen. by $\{\prod_\alpha E_\alpha \mid E_\alpha \in \mathcal{M}_\alpha\}$)

Example: $\mathbb{R}^n = \prod_{j=1}^{n} \mathbb{R}$

gen. by $\pi_j^{-1}(E)$, $E \in \mathcal{B}_{\mathbb{R}}$

$\pi_j : (x_1, \dots, x_n) \longmapsto x_j$

Prop: $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{j=1}^{n} \mathcal{B}_{\mathbb{R}}$

gen. by open sets in $\mathbb{R}^n$

(exercise / text)

Fact: $f : X \longrightarrow Y = \prod_{\alpha \in A} Y_\alpha$ is $(\mathcal{M}, \bigotimes_{\alpha \in A} \mathcal{N}_\alpha)$-measurable

$\Longleftrightarrow f_\alpha := \pi_\alpha \circ f : X \to Y_\alpha$ is measurable $\forall \alpha$

(exercise/text)

Rem: $f: X \to \overline{\mathbb{R}} = [-\infty, \infty]$ is measurable if it is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-meas, $\mathcal{B}_{\overline{\mathbb{R}}} = \{E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$

Prop: $(X, \mathcal{M})$ measurable space

1. $f: X \to \mathbb{C}$ is meas $\iff \operatorname{Re} f, \operatorname{Im} f$ meas.
2. $f, g: X \to \mathbb{C}$ meas $\Rightarrow f+g, f \cdot g$ meas.
3. $f_j: X \to \overline{\mathbb{R}}$ meas. $\Rightarrow \sup_j f_j, \inf_j f_j, \overline{\lim_{j\to\infty}} f_j, \underline{\lim_{j\to\infty}} f_j$ meas.
4. $f_j: X \to \mathbb{C}$ meas. $\Rightarrow \lim_{j\to\infty} f_j$ (if $\exists$) is meas.

pointwise

Proof (sketch):

1. $\mathcal{B}_{\mathbb{C}} = \mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$

+ "fact" above ✓

2. $F : X \to \mathbb{C} \times \mathbb{C}$

$x \longmapsto (f(x), g(x))$

meas. (by above "Fact")

+ comp. is meas. of meas.

$\left.\begin{array}{l} \phi(z,w) := z + w \\ \psi(z,w) := zw \end{array}\right\}$ meas. (since continuous)

✓

3, 4: see text.