Regularity of Lebesgue Measure (1.5)

G_δ = c'ble $\bigcap$ of opens

F_σ = c'ble $\bigcup$ of closeds

$$\mathcal{P}(\mathbb{R}) \supsetneq \mathcal{L} \supsetneq \mathcal{B}_{\mathbb{R}} \supsetneq \text{open sets, closed sets}$$

(Lebesgue) measurable

Thm: $E \subset \mathbb{R}$

(a) $E \in \mathcal{L}$

(b) $\iff \forall\, \varepsilon > 0,\ \exists\ U^{\text{open}} \supset E$ s.t. $m^*(U \setminus E) \le \varepsilon$

(c) $\iff$ " " " $F^{\text{closed}} \subset E$ " $m^*(E \setminus F) \le \varepsilon$

(d) $\iff \exists\ G_\delta$-set $V \supset E$ s.t. $E = V \setminus N_1$, N_1 null

(e) $\iff \exists\ F_\sigma$-set $H \subset E$ s.t. $E = H \cup N_2$, N_2 null

<u>Lemma:</u> $m^*(E) = \inf\left\{ \sum_{j=1}^{\infty}(b_j - a_j) \;\middle|\; E \subset \bigcup_{j=1}^{\infty}(a_j, b_j) \right\}$

Pf: exercise/text

<u>Proof:</u> (a)$\Rightarrow$(b): given $\varepsilon > 0$, $\exists\, \overset{\text{open}}{U} = \bigcup_{j=1}^{\infty} I_j \supset E$ s.t. $m(E) + \varepsilon \geq \sum_{j=1}^{\infty} m(I_j) \geq m(U)$

- $m(U) \overset{E \in \mathcal{L}}{=} m(\overbrace{U \cap E}^{E}) + m(\underbrace{U \cap E^c}_{U \setminus E}) \leq m(E) + \varepsilon$

$\Rightarrow m(U \setminus E) \leq \varepsilon$ if $m(E) < \infty$

- if $m(E) = \infty$, $E_j := E \cap [j, j-1)$, $j \in \mathbb{Z}$

$\Rightarrow \ \exists \ U_j^{\text{open}} \supset E_j$ s.t. $m(U_j \setminus E_j) \leq \frac{\varepsilon}{2^{|j|}}$

. so $U = \bigcup_{j=1}^{\infty} U_j \supset E$, and

$$U \setminus E = U \cap \Big(\bigcup_{j \in \mathbb{Z}} E_j\Big)^c = U \cap \bigcap_j E_j^c$$

$$= \bigcup_{k \in \mathbb{Z}} \Big(U_k \cap \underbrace{\bigcap_j E_j^c}_{\subset E_k^c}\Big) \subset \bigcup_{k \in \mathbb{Z}} U_k \setminus E_k$$

$$\Rightarrow m(U \setminus E) \leq \sum_{k \in \mathbb{Z}} m(U_k \setminus E_k) \leq \sum_{k \in \mathbb{Z}} \frac{\varepsilon}{2^{|k|}} = 3\varepsilon \quad \checkmark$$

(a) $\Rightarrow$ (c) : consider E^c, use (a) $\Rightarrow$ (b)

(b) $\Rightarrow$ (d) : $\forall\ j = 1, 2, 3, \dots$ $\exists\ U_j \overset{\text{open}}{\supset} E$ s.t. $m^*(U_j \setminus E) \le \frac{1}{j}$

· set : $V := \bigcap_{j=1}^{\infty} U_j \supset E$, ($G_\delta$ -set)

· $N_1 := V \setminus E \Rightarrow E = V \setminus N_1$

$\vdots$ $N_1 \subset U_j \setminus E\ \forall j \Rightarrow m^*(N_1) \le m^*(U_j \setminus E) \le \frac{1}{j} \xrightarrow[j \to \infty]{} 0$

$\Rightarrow m^*(N_1) = 0.$

(c) $\Rightarrow$ (e) similar

· (d) or (e) $\Rightarrow$ (a) : G_δ, F_σ, null sets $\in \mathcal{L}$

Ex: Cantor set

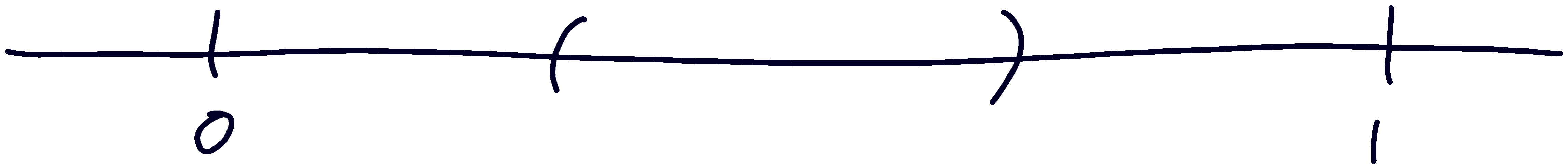
- "measure" "size" of sets
  - (Lebesgue) measure theoretically
  - set theoretically (cardinality)
  - topologically (open, dense ?)

Prop:

a) $m(C) = 1 \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdots = 0$

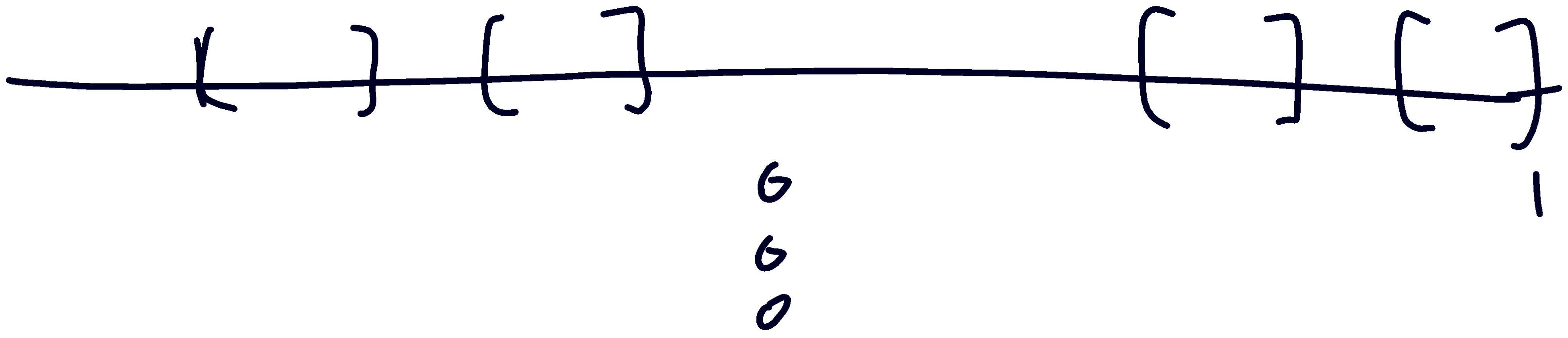
b) compact, nowhere dense, no isolated pts, completely disconnected

c) $\mathrm{card}(C) = \mathrm{card}(\mathbb{R})$ (– see text – ternary expansions)



$K_0 = 0,1]$

$K_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$



$K_2 = - - - \ - .$

$C := \bigcap_{j=1}^{\infty} K_j$