The plot so far . . .

- “length” $m_0$, on the algebra $\mathcal{A} = \{\text{finite disjoint unions of } h\text{-intervals} \}$, is a

premeasure: \[
\begin{cases}
m_0(\emptyset) = 0; \\
\{A_j\}_{j=1}^\infty \subset \mathcal{A} \text{ disjoint, } \bigcup_{j=1}^\infty A_j \in \mathcal{A} & \implies m_0(\bigcup_{j=1}^\infty A_j) = \sum_{j=1}^\infty m_0(A_j) \\
& \text{– countably additive, but on not enough sets}
\end{cases}
\]

- we try to measure all subsets via (countable!) coverings by sets in $\mathcal{A}$:

\[
m^*(E) := \inf \{ \sum_{j=1}^\infty m_0(A_j) \mid E \subset \bigcup_{j=1}^\infty A_j, \ A_j \in \mathcal{A} \}, \quad \text{producing an}
\]

outer measure: \[
\begin{cases}
m^*(\emptyset) = 0; \\
E \subset F & \implies m^*(E) \leq m^*(F); \\
m^*(\bigcup_{j=1}^\infty E_j) \leq \sum_{j=1}^\infty m^*(E_j)
\end{cases}
\]

– measures all sets, but is merely subadditive

- to recover additivity, we restrict to subsets $A \subset \mathbb{R}$ which are $m^*$-measurable: \[
\forall E \subset \mathbb{R}, \quad m^*(E) = m^*(E \cap A) + m^*(E \cap A^c)
\]

- Carathéodory’s Theorem: $\mathcal{M} := \{m^* – \text{measurable sets}\}$ is a $\sigma$-algebra

\[
m^* \upharpoonright \mathcal{M} \text{ is a (complete) measure}
\]

- remaining task: $m^* \upharpoonright \mathcal{M}$ extends “length”: $\mathcal{A} \subset \mathcal{M}$ and $m^* \upharpoonright \mathcal{A} \equiv m_0 \upharpoonright \mathcal{A}$
premeasure $\mu_0 : A \rightarrow [0, \infty]$ s.t. $\mu_0(\emptyset) = 0$

outer measure: $E \subset X$

$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(E_j) \mid E \subset \bigcup_{j=1}^{\infty} E_j, E_j \in A \right\}$

Prop: 1. $A \in \mathcal{A} \Rightarrow A$ $\mu^*$-measurable

Prop: 2. $\mu^*|_\mathcal{A} = \mu_0|_\mathcal{A}$

Proof: 2. see text

$A \cap M(A) \subset \mu^*$-meas $\mathcal{C}_P(X)$
1. Let $A \in \mathcal{F}$, $E \subset X$. Let $\varepsilon > 0$. 

$\exists \{A_j\}_{j=1}^{\infty} \subset A$ s.t. $\mathcal{N}^*(E) + \varepsilon \geq \sum_{j=1}^{\infty} \mathcal{N}_0(A_j)$

$\Rightarrow \mathcal{N}^*(E) + \varepsilon \geq \mathcal{N}^*(E \cap A) + \mathcal{N}^*(E \cap A^c)$

$\Rightarrow \varepsilon \rightarrow 0 \quad (\leq \text{for free}) \Rightarrow A$ is $\mathcal{N}^*$-measurable
Lebesgue Measure on $\mathbb{R}$ (1.5)

- $X = \mathbb{R}$, $\mathcal{N}_0 = \mathcal{M}_0 = \text{lengths of } h\text{-intervals}$ on $\mathcal{A} = \{ \text{finite, disjoint } U\text{'s of } h\text{-intervals} \}$

  - $m^*(E) = \inf \{ \sum_{j=1}^{\infty} m_0(I_j) \mid E \subset \bigcup_{j=1}^{\infty} I_j \}$ [Lebesgue outer measure]

  - $\mathcal{L} = m^*$-measurable sets
  - $m = m^*|_{\mathcal{L}} = \text{Lebesgue measure} \ (\text{complete})$
Remarks: 1. $m$ is a Borel measure, since

\[ A \subset B \subset C \subset P(\mathbb{R}) \]

\[ A \neq B \neq C \neq P(\mathbb{R}) \]

\[ M(A) \]

2. $m$ is the unique Borel measure s.t.

\[ m((a,b)) = b - a \]

\[ Pf: \text{(see text)} \] if $\mu_0$ is $\sigma$-finite on $\mathbb{R},$

then $N$ is the ! measure on $M(A)$ extending $\mu_0$
3. can make the same construction, starting from
\[ \text{M}_F(a,b] = F(b) - F(a) \quad (m \Rightarrow F(x) = x) \]

where \( F \) is increasing \((x \leq y \Rightarrow f(x) \leq f(y))\)
- right continuous (exercise: why?)

\[ \text{Lebesgue-Stieltjes measure, } M_F, \text{ on } \mathbb{R} \]

4. Prop: \( \mu \) a Borel meas. on \( \mathbb{R} \), finite on bounded sets:
\[ F(\xi) := \begin{cases} \mu((0, \xi]) & x > 0 \\ 0 & x = 0 \\ -\mu((\xi, 0]) & x < 0 \end{cases} \]
is incr., right cont.; Proof: text.
5. $M$ is translation invariant: $m(E+s) = m(E)$

Proof: see text (e.g. $m_s(E) := m(E+s)$ is a Borel measure, so must agree with $m(E)$)