The plot so far ...

- "length" m_0, on the algebra $\mathcal{A} = \{\text{finite disjoint unions of } h\text{-intervals}\}$, is a

premeasure: $\begin{cases} m_0(\emptyset) = 0; \\ \{A_j\}_{j=1}^{\infty} \subset \mathcal{A} \text{ disjoint}, \bigcup_{j=1}^{\infty} A_j \in \mathcal{A} \implies m_0(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} m_0(A_j) \end{cases}$

– countably additive, but on not enough sets

- we *try* to measure *all* subsets via (countable!) coverings by sets in $\mathcal{A}$:

$m^*(E) := \inf \{ \sum_{j=1}^{\infty} m_0(A_j) \mid E \subset \cup_{j=1}^{\infty} A_j, \ \ A_j \in \mathcal{A} \}$, producing an

outer measure: $\begin{cases} m^*(\emptyset) = 0; \\ E \subset F \implies m^*(E) \leq m^*(F); \\ m^*(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} m^*(E_j) \end{cases}$

– measures all sets, but is merely *subadditive*

- to recover additivity, we restrict to subsets $A \subset \mathbb{R}$ which are

m^***-measurable**: $\forall\, E \subset \mathbb{R}, \quad m^*(E) = m^*(E \cap A) + m^*(E \cap A^c)$

- **Carathéodory's Theorem**: $\mathcal{M} := \{m^*-\text{ measurable sets}\}$ is a σ-algebra

  $m^* \restriction_{\mathcal{M}}$ is a (complete) measure

- remaining task: $m^* \restriction_{\mathcal{M}}$ *extends* "length": $\mathcal{A} \subset \mathcal{M}$ and $m^* \restriction_{\mathcal{A}} \equiv m_0 \restriction_{\mathcal{A}}$

- premeasure $\underset{\text{alg.}}{\mu_0 : \mathcal{A}} \to [0,\infty]$: $\begin{cases} \mu_0(\emptyset) = 0 \\ \mu_0\Big(\underbrace{\bigcup_{j=1}^{\infty} A_j}_{\text{disjoint, } \in \mathcal{A}}\Big) = \sum_{j=1}^{\infty} \mu_0(A_j) \\ A_j \in \mathcal{A} \end{cases}$

- outer measure: $E \subset X$

$$\mu^*(E) = \inf\left\{ \sum_{j=1}^{\infty} \mu_0(E_j) \;\middle|\; E \subset \bigcup_{j=1}^{\infty} E_j,\ E_j \in \mathcal{A} \right\}$$

<u>Prop:</u> 1. $A \in \mathcal{A} \Rightarrow A$ μ^*-measurable ✓

2. $\mu^*\restriction_{\mathcal{A}} = \mu_0\restriction_{\mathcal{A}}$ ✓

<u>Proof:</u> 2. see text

$$\left(\mathcal{A} \subset \mathcal{M}(\mathcal{A}) \subset \mu^*\text{-meas} \subset \mathcal{P}(X) \right)$$

1. Let $A \in \mathcal{A}$, $E \subset X$. Let $\varepsilon > 0$.

$\exists \{A_j\}_{j=1}^{\infty} \subset \mathcal{A}$ s.t. $\mu^*(E) + \varepsilon \geq \sum_{j=1}^{\infty} \mu_0(\overbrace{A_j}^{(A_j \cap A) \cup (A_j \cap A^c)})$

& $E \subset \bigcup_{j=1}^{\infty} A_j$

$$= \sum_{j=1}^{\infty} \mu_0(\underbrace{A_j \cap A}_{\substack{\text{cover } E\cap A \\ (\text{by } \mathcal{A})}}) + \sum_{j=1}^{\infty} \mu_0(\underbrace{A_j \cap A^c}_{\substack{\text{cover } E\cap A^c \\ (\text{by } \mathcal{A})}})$$

$$\geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

$\Rightarrow \mu^*(E) + \varepsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$

$\underset{(\varepsilon \to 0)}{\Longrightarrow} \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$

($\leq$ for free)

$\Rightarrow A$ is μ^*-measurable ✓

Lebesgue Measure on $\mathbb{R}$ (1.5)

- $X = \mathbb{R}$, $\mu_0 = m_0 =$ lengths of h-intervals on $\mathcal{A} = \left\{ \begin{array}{l} \text{finite, disjoint} \\ \cup\text{'s of} \\ \text{h-intervals} \end{array} \right\}$
- $m^*(E) = \inf \left\{ \sum_{j=1}^{\infty} m_0(I_j) \;\middle|\; E \subset \bigcup_{j=1}^{\infty} I_j \right\}$ Lebesgue outer measure

  (I_j: h-intervals)
- $\mathcal{L} = m^*$-meas. sets $=$ Lebesgue measurable sets
- $m = m^*\big|_{\mathcal{L}} =$ Lebesgue measure (complete)

Remarks: 1. m is a Borel measure, since

$$\mathcal{A} \underset{\neq}{\subset} \underset{\substack{\shortparallel \\ \mathcal{M}(\mathcal{A})}}{\mathcal{B}_{\mathbb{R}}} \underset{\neq}{\subset} \mathcal{L} \underset{\neq}{\subset} \mathcal{P}(\mathbb{R})$$

2. m is the unique Borel measure s.t.

$$m((a,b]) = b-a$$

Pf: (see text) if μ_0 is σ-finite on $\mathcal{A}$, then μ is the ! measure on $\mathcal{M}(\mathcal{A})$ extending μ_0

3. can make the same construction, starting from $m_F(a,b] = F(b) - F(a)$ $\quad$ ($m \leftrightarrow \underline{F(x) = x}$)

where F is
- increasing ($x \le y \Rightarrow f(x) \le f(y)$)
- right continuous (exercise: why?)

$\longrightarrow$ <u>Lebesgue–Stieltjes measure</u>, m_F, on $\mathcal{B}_{\mathbb{R}}$

4. <u>Prop</u>: μ a Borel meas. on $\mathbb{R}$, finite on bounded sets:

$$F(x) := \begin{cases} \mu((0,x]) & x > 0 \\ 0 & x = 0 \\ -\mu((x,0]) & x < 0 \end{cases}$$

is incr., right cont., and $\mu = m_F$ $\quad$ Proof: text.

5. m is
- translation invariant: $m(E+s) = m(E)$ $s \in \mathbb{R}$
- dilation " $m(rE) = |r| m(E)$

<u>Proof</u>: see text (eg: $m_s(E) := m(E+s)$ is a Borel measure, so must agree with $m(E)$)