The plot so far . . .

- "length" m_0, on the algebra $\mathcal{A} = \{\text{finite disjoint unions of } h\text{-intervals}\}$, is a

premeasure: $\begin{cases} m_0(\emptyset) = 0; \\ \{A_j\}_{j=1}^{\infty} \subset \mathcal{A} \text{ disjoint}, \bigcup_{j=1}^{\infty} A_j \in \mathcal{A} \implies m_0(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} m_0(A_j) \end{cases}$

  – countably additive, but on not enough sets

- we *try* to measure *all* subsets via (countable!) coverings by sets in $\mathcal{A}$:

  $m^*(E) := \inf \{ \sum_{j=1}^{\infty} m_0(A_j) \mid E \subset \cup_{j=1}^{\infty} A_j, \;\; A_j \in \mathcal{A} \}$, producing an

outer measure: $\begin{cases} m^*(\emptyset) = 0; \\ E \subset F \implies m^*(E) \leq m^*(F); \\ m^*(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} m^*(E_j) \end{cases}$

  – measures all sets, but is merely *subadditive*

- to recover additivity, we restrict to subsets $A \subset \mathbb{R}$ which are

m^*-**measurable**: $\forall\, E \subset \mathbb{R}, \quad m^*(E) = m^*(E \cap A) + m^*(E \cap A^c)$

- remaining tasks:
  - m^*-measurable sets are a σ-algebra (containing $\mathcal{A}$),
  - on which m^* is a measure, extending "length" m_0

Carathéodory's Theorem (1.4)

- outer measure $\mu^*: \mathcal{P}(X) \to [0, \infty]$
  - $\mu^*(\phi) = 0$
  - $A \subset B \Rightarrow \mu^*(A) \le \mu^*(B)$
  - $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \le \sum_{j=1}^{\infty} \mu^*(A_j)$

- $\mathcal{M} :=$ μ^*-measurable sets: $A \subset X$ s.t.

$$\forall E \subset X, \quad \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

- "$\le$" is automatic by
- need to show "$\ge$"

$$\boxed{\text{Thm: } \mathcal{M} \text{ is a } \sigma\text{-algebra, and } \mu^*\restriction_{\mathcal{M}} \text{ is a (complete) measure.}}$$

Proof: $(\phi \in \mathcal{M})$

1. $\mathcal{M}$ is an algebra: i.e closed under

- complement ✓
- union: $A, B \in \mathcal{M}$:

$$\mu^*(E) = \mu^*(E\cap A) + \mu^*(E\cap A^c)$$

$$= \underbrace{\mu^*(E\cap A\cap B) + \mu^*(E\cap A\cap B^c) + \mu^*(E\cap A^c\cap B)}_{\geq\, \mu^*(E\cap(A\cup B))} + \mu^*(\underbrace{E\cap A^c\cap B^c}_{E\cap(A\cup B)^c})$$

since
- $E\cap(A\cup B) \subset \left\{\begin{array}{l} E\cap A\cap B \\ \cup (E\cap A\cap B^c) \\ \cup (E\cap A^c\cap B)\end{array}\right.$
- subadditivity

$$\Longrightarrow A\cup B \in \mathcal{M} \quad ✓$$

2. μ^* is finitely additive on $\mathcal{M}$: $A, B \in \mathcal{M}$, disjoint

$$\Rightarrow \mu^*(A \cup B) = \mu^*(\underbrace{A \cup B \cap A}_{A}) + \mu^*(\underbrace{A \cup B \cap A^c}_{B}) \checkmark$$

3. $\mathcal{M}$ is closed under countable disjoint unions (hence a σ-algebra) and μ^* is countably additive on $\mathcal{M}$: $\{A_j\}_{j=1}^{\infty} \subset \mathcal{M}$ disjoint

. set $B_n := \bigcup_{j=1}^{n} A_j$, $B = \bigcup_{j=1}^{\infty} A_j$, let $E \subset X$:

$$\mu^*(E \cap B_n) = \mu^*(\underbrace{E \cap B_n \cap A_n}_{E \cap A_n}) + \mu^*(\underbrace{E \cap B_n \cap A_n^c}_{E \cap B_{n-1}})$$

$$= \mu^*(E \cap A_n) + \mu^*(E \cap A_{n-1}) + \mu^*(E \cap B_{n-2}) = \cdots$$

$$= \sum_{j=1}^{n} \mu^*(E \cap A_j)$$

- $\mu^*(E) = \underbrace{\mu^*(E \cap B_n)}_{= \sum_{j=1}^{n} \mu^*(E \cap A_j)} + \underbrace{\mu^*(E \cap B_n^c)}_{\geq \mu^*(E \cap B^c)}$ $\quad B_n^c \supset B^c$ (monotonicity)

- take $n \to \infty$ $\Rightarrow \mu^*(E) \geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*(E \cap B^c)$

$\Rightarrow B \in \mathcal{M}$ ✓ $\quad \geq \mu^*(E \cap \underbrace{\bigcup_{j=1}^{\infty} A_j}_{B}) + \mu^*(E \cap B^c)$ (subadditivity)

- finally, take $E = B$

$\Rightarrow \mu^*(B) = \sum_{j=1}^{n \to \infty} \mu^*(A_j) + \cancel{\mu^*(\emptyset)} \to 0$

$\Rightarrow$ additivity ✓

$\Rightarrow \mu^*\restriction_{\mathcal{M}}$ is a measure

- completeness: if $N \in \mathcal{M}$, $\mu^*(N) = 0$ and

$$F \subset N \implies F \in \mathcal{M}$$

(exercise).

- note: $\mu^*(F) = 0$.

- next:
  - $\mathcal{A} \subset m^*$-measurable sets
  - $m^*\restriction_{\mathcal{A}} = m_0\restriction_{\mathcal{A}}$