The plot so far . . .

• “length” $m_0$, on the algebra $\mathcal{A} = \{\text{finite disjoint unions of } h\text{-intervals}\}$, is a premeasure:

$$m_0(\emptyset) = 0; \quad \{A_j\}_{j=1}^{\infty} \subset \mathcal{A} \text{ disjoint, } \bigcup_{j=1}^{\infty} A_j \in \mathcal{A} \implies m_0 \left( \bigcup_{j=1}^{\infty} A_j \right) = \sum_{j=1}^{\infty} m_0(A_j)$$

– countably additive, but on not enough sets

• we try to measure all subsets via (countable!) coverings by sets in $\mathcal{A}$:

$$m^*(E) := \inf \{ \sum_{j=1}^{\infty} m_0(A_j) \mid E \subset \bigcup_{j=1}^{\infty} A_j, \ A_j \in \mathcal{A} \}, \quad \text{producing an outer measure:}$$

$$m^*(\emptyset) = 0; \quad E \subset F \implies m^*(E) \leq m^*(F); \quad m^* \left( \bigcup_{j=1}^{\infty} E_j \right) \leq \sum_{j=1}^{\infty} m^*(E_j)$$

– measures all sets, but is merely subadditive

• to recover additivity, we restrict to subsets $A \subset \mathbb{R}$ which are $m^*$-measurable:

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^c)$$

• remaining tasks:

– $m^*$-measurable sets are a $\sigma$-algebra (containing $\mathcal{A}$),

– on which $m^*$ is a measure, extending “length” $m_0$
Carathéodory's Theorem (1.4)
- outer measure \( \mu^* : P(X) \to [0, \infty] \)
- \( M^* \) measurable sets: \( A \subseteq X \) s.t.
  \[ \forall E \in X, \quad \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \]

Thm: \( M^* \) is a \( \sigma \)-algebra, and \( \mu^* | M \) is a measure
  (complete)
Proof: 1. \( M \) is an algebra: i.e. closed under

- complement \( \checkmark \)
- union: \( A, B \in M \):

\[
N^*(E) = N^*(E \cap A) + N^*(E \cap A^c)
= N^*(E \cap A \cap B) + N^*(E \cap A \cap B^c) + N^*(E \cap A \cap B^c) + N^*(E \cap A \cap B)
\]

since \( E \cap (A \cup B) \subseteq E \cap A \cap B \) \( \cup \) subadditivity

\[
\supseteq N^*(E \cap (A \cup B))
\]

\( \Rightarrow A \cup B \in M \) \( \checkmark \)
2. \( \mu^* \) is finitely additive on \( M \): \( A, B \in M \), disjoint

\[ \Rightarrow \mu^*(A \cup B) = \mu^*(A \cup B \cap A) + \mu^*(A \cup B \cap A^c) \]

3. \( M \) is closed under countable disjoint unions (\( \sigma \)-algebra) and \( \mu^* \) is countably additive on \( M \): \( \{A_j\}_{j=1}^\infty \in M \) disjoint

Set \( B_n = \bigcup_{j=1}^n A_j \), \( B = \bigcup_{j=1}^\infty A_j \). Let \( E \in X \):

\[ \mu^*(E \cap B_n) = \mu^*(E \cap B_n \cap A_j) + \mu^*(E \cap B_n \cap A_j^c) \]

\[ = \mu^*(E \cap A_j) + \mu^*(E \cap A_j^c) + \mu^*(E \cap B_n) = 0 \text{ if } E \in E \]

\[ \mu^* \left( \bigcup_{j=1}^\infty E \cap A_j \right) = \sum_{j=1}^\infty \mu^*(E \cap A_j) \]
\[ \nu^*(E) = \nu^*(E \cap B_n) + \nu^*(E \cap B_n^c) \geq \nu^*(E \cap B) \quad (\text{monotonicity}) \]

take \( n \to \infty \Rightarrow \nu^*(E) \geq \sum_{j=1}^{\infty} \nu^*(E \cap A_j) + \nu^*(E \cap B^c) \]

\( \Rightarrow B \in M \checkmark \)

finally, take \( E = B \Rightarrow \nu^*(B) = \sum_{j=1}^{\infty} \nu^*(A_j) + \nu^*(B^c) \rightarrow 0 \)

\( \Rightarrow \text{additivity } \checkmark \)
\[ \Rightarrow \nu^* \]

is a measure

\[ M \]

completeness: if \( N \in M \), \( \nu^*(N)=0 \) and \( F \subset N \Rightarrow F \in M \)

(exercise).

\[ \text{note: } \nu^*(\emptyset)=0. \]

next:

\[ A \subset m^*-\text{measurable sets} \]

\[ m^* \upharpoonright A = m_0 \upharpoonright A. \]