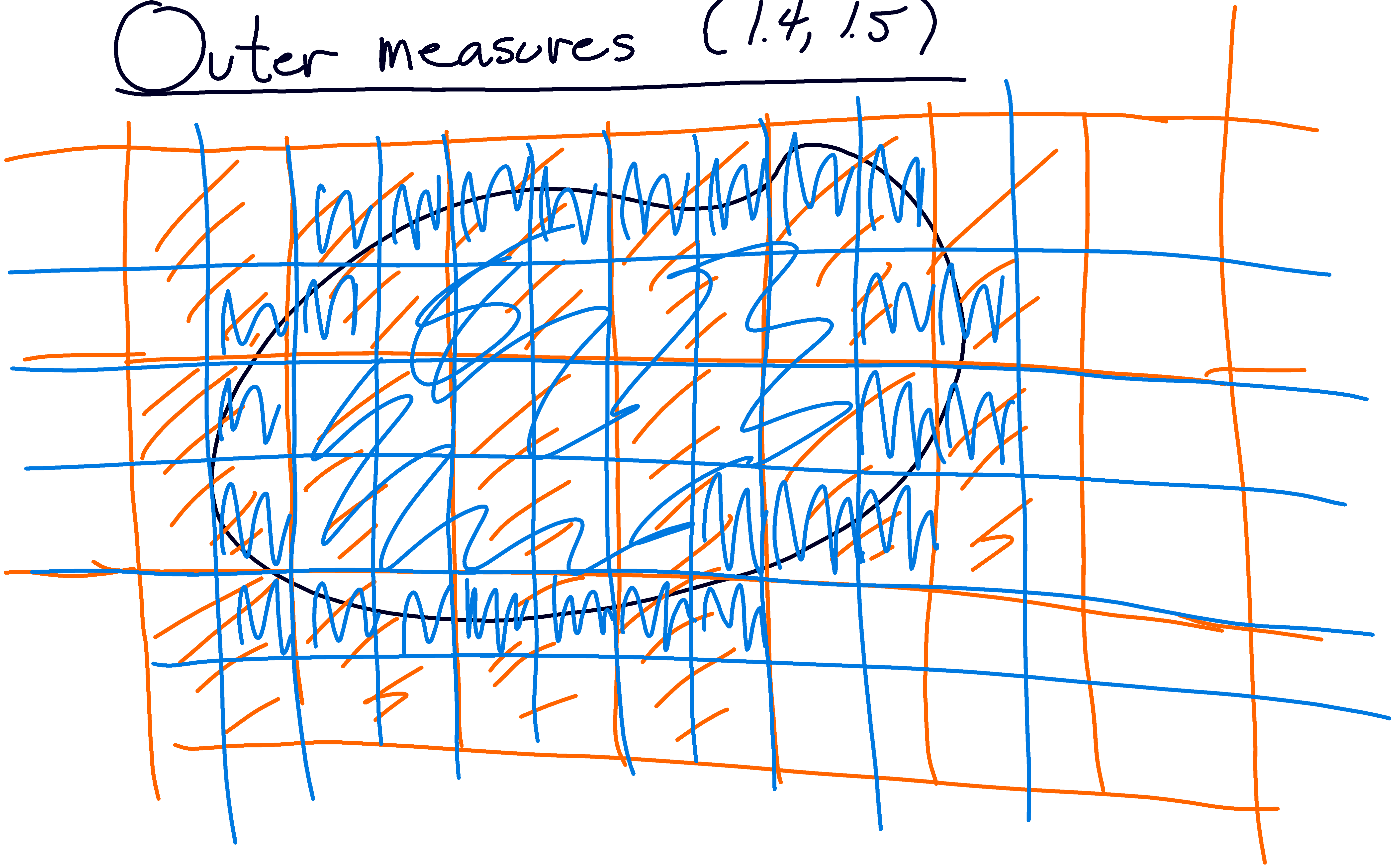
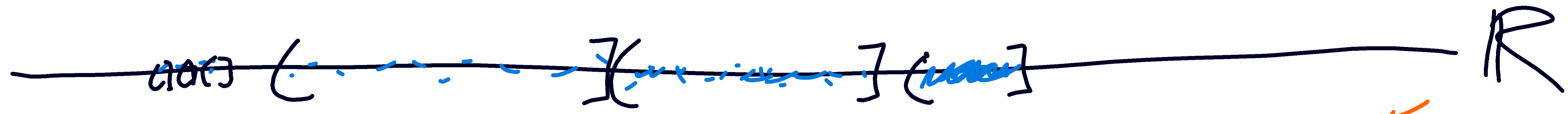
Outer measures (1.4, 1.5)





$E \subset \mathbb{R}$ $\qquad m^*(E) = \inf\left\{ \sum_{j=1}^{\infty} m_0(I_j) \;\middle|\; E \subset \bigcup_{j=1}^{\infty} I_j \right\}$

length $\qquad$ h-intervals

"Lebesgue outermeasure"

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in general: given any $\mathcal{E} \subset \mathcal{P}(X)$ (with $\emptyset, X \in \mathcal{E}$)

and $\nu_0 : \mathcal{E} \longrightarrow [0, \infty]$, with $\nu_0(\emptyset) = 0$

define $\nu^* : \mathcal{P}(X) \longrightarrow [0, \infty]$ by (1)

$$\boxed{\nu^*(E) := \inf\left\{ \sum_{j=1}^{\infty} \nu_0(E_j) \;\middle|\; E \subset \bigcup_{j=1}^{\infty} E_j,\ E_j \in \mathcal{E} \right\}}$$

$E \subset \bigcup_{j=1}^{\infty} X$

Prop: (1) produces an $\boxed{\text{outer measure}}$:

1. $\mu^*(\phi) = 0$
2. $A \subset B \Rightarrow \mu^*(A) \le \mu^*(B)$ "monotonicity"
3. $\mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) \le \sum_{j=1}^{\infty} \mu^*(E_j)$ "sub additivity"

Proof: 1. $\phi \subset \bigcup_{j=1}^{\infty} \phi$ ✓

$\Rightarrow \mu^*(\phi) \le \sum_{j=1}^{\infty} \mu_0(\phi) = 0$

2. $A \subset B$

$$\left\{ \{E_j\}_{j=1}^{\infty} \subset \mathcal{E} \;\middle|\; A \subset \bigcup_{j=1}^{\infty} E_j \right\}$$

$$\cup$$

$$\left\{ \{E_j\}_{j=1}^{\infty} \subset \mathcal{E} \;\middle|\; B \subset \bigcup_{j=1}^{\infty} E_j \right\} \Longrightarrow \mu^*(A) \le \mu^*(B)$$ ✓

Let $\varepsilon > 0$

$\exists \quad \frac{\varepsilon}{2^j} + \mu^*(E_j) \geq \sum_{k=1}^{\infty} \mu_0(A_j^k) \quad \Big| \quad E_j \subset \bigcup_{k=1}^{\infty} A_j^k \quad \exists$

$A_j^k \in \mathcal{E}$

- $\bigcup_{j=1}^{\infty} E_j \subset \bigcup_{j,k=1}^{\infty} A_j^k \Rightarrow \mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j,k=1}^{\infty} \mu_0(A_j^k)$

$$\leq \sum_{j=1}^{\infty} \left(\frac{\varepsilon}{2^j} + \mu^*(E_j) \right) = \varepsilon + \sum_{j=1}^{\infty} \mu^*(E_j)$$

- since ε arbitrary

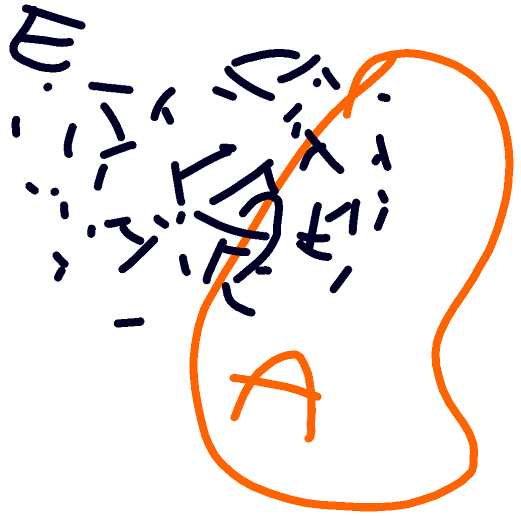
✓ $\Rightarrow \mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu^*(E_j)$

μ^* :
- defined $\forall\, \mathcal{P}(X)$ ✓
- not additive ✗ → remedy: remove some "bad" sets

Def: μ^* an outer measure on X.

A set $A \subset X$ are ***μ^*-measurable*** if:

$$\forall E \subset X, \quad \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$



intuition: $A \subset E$, "nice"

$$\mu^*(A) = \underbrace{\mu^*(E) - \mu^*(E \setminus A)}_{\text{"inner measure"}}$$

"outer = inner"