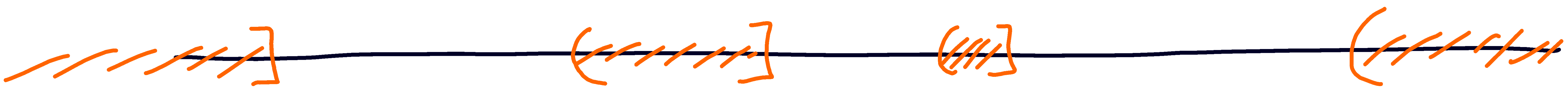
Premeasures (1.5)



$\mathbb{R}$

$$\mathcal{A} := \left\{ \begin{array}{l} \text{finite disjoint unions of "h-intervals":} \\ (a,b],\ (a,\infty),\ \emptyset,\ -\infty \le a < b < \infty \end{array} \right\}$$

Prop: $\mathcal{A}$ is an algebra ($A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$, $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$)

Pf:
- intersection of 2 h-intervals is an intervals
- complement of an h-interval is a disjoint union of h-intervals
- Folland, Prop. 1.7

$\blacksquare$

• the "length" of sets $\in \mathcal{A}$:

$$m_0 : \mathcal{A} \longrightarrow [0, \infty]$$

$$\left\{ \begin{array}{lcl} \text{disjoint} \rightarrow \bigcup_{j=1}^{n} [a_j, b_j] & \longmapsto & \sum_{j=1}^{n} (b_j - a_j) \\ \phi & \longmapsto & 0 \\ \text{any } \bigcup \text{ with } \infty^{\text{te}} \text{ interval} & \longmapsto & \infty \end{array} \right\}$$

Thm: 1. m_0 is well-defined

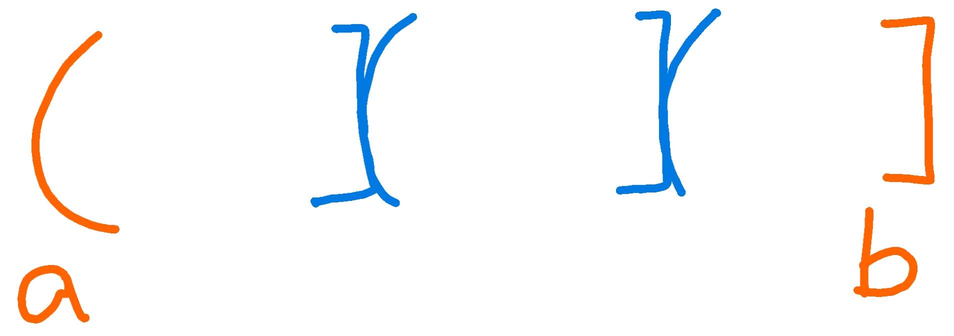
2. m_0 is a premeasure on algebra $\mathcal{A}$:

Def:

$m_0 : \mathcal{A} \longrightarrow [0, \infty]$

1. $m_0(\phi) = 0$
2. if $\{A_j\}_{j=1}^{\infty} \subset \mathcal{A}$, disjoint, and $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$

$\Rightarrow m_0\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} m_0(A_j)$

Pf: 1. "problem": 

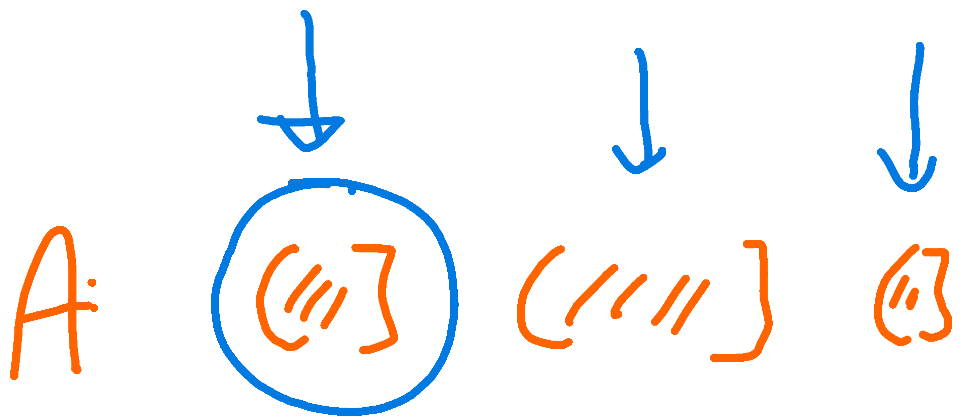
(for finite, disjoint unions of h-intervals)

$\longrightarrow$ see text

- note: m_0 is finitely additive

2. • suppose $A = \bigcup_{j=1}^{\infty} I_j$ ← h-intervals

$A \supset A$ ← disjoint



- goal: show $\boxed{m_0(A) = \sum_{j=1}^{\infty} m_0(I_j)}$

- may assume $A = I$, an h-interval

(by • taking subsequences of $\{I_j\}$
• finite additivity)

- take $I = (a, b]$, $-\infty < a < b < \infty$

(for ∞-intervals, see text/exercise)

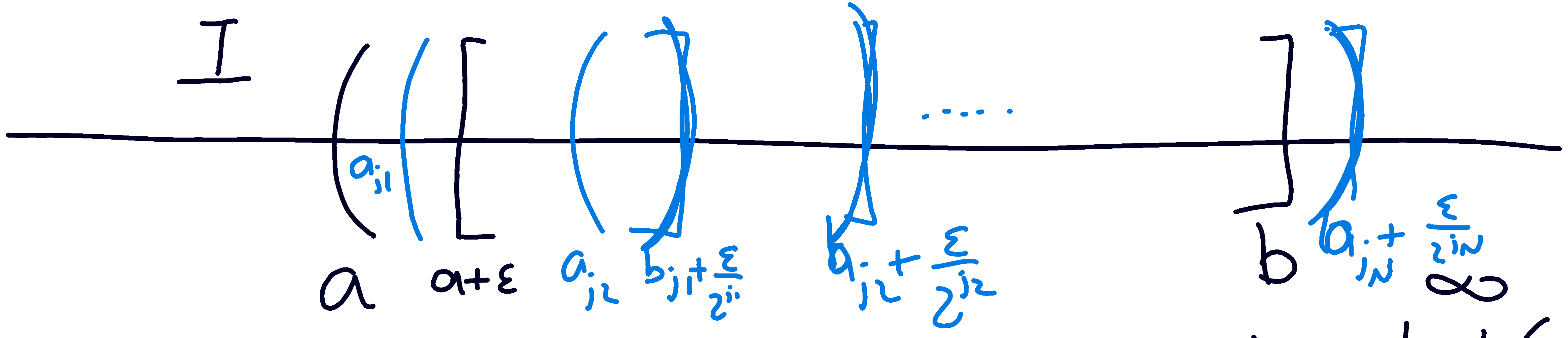
- $$I = \underbrace{\bigcup_{j=1}^{n} I_j}_{\in \mathcal{A}} \underset{\text{disjoint}}{\cup} \underbrace{\left(I \setminus \bigcup_{j=1}^{n} I_j \right)}_{\in \mathcal{A}}$$

$$\Rightarrow m_0(I) = m_0\left(\bigcup_{j=1}^{n} I_j\right) + \underbrace{m_0\left(I \setminus \bigcup_{j=1}^{n} I_j\right)}_{\geq 0}$$

$$\geq \sum_{j=1}^{n} m_0(I_j)$$

($\frac{1}{2}$-way!)

- take $n \to \infty$ $\Longrightarrow$ $m_0(I) \geq \sum_{j=1}^{\infty} m_0(I_j)$



$$I = \bigcup_{j=1}^{\infty} (a_j, b_j]$$

· let $\varepsilon > 0$. $[a+\varepsilon, b]$ is covered by $\underbrace{\bigcup_{j=1}^{\infty} \left(a_j, b_j + \frac{\varepsilon}{2^j}\right)}_{\text{"open cover"}}$

compact

$$\Rightarrow [a+\varepsilon, b] \subset \bigcup_{k=1}^{N} \left(a_{j_k}, b_{j_k} + \frac{\varepsilon}{2^{j_k}}\right)$$

s.t. (relabelling)

· $a_{j_1} < a_{j_2} < \cdots < a_{j_N}$

· $b_{j_k} \in \left(a_{j_{k+1}}, b_{j_{k+1}} + \frac{\varepsilon}{2^{j_{k+1}}}\right)$

$$\Rightarrow m_0(a,b] = b-a \le b_{j_N} + \frac{\varepsilon}{2^{j_N+1}} \underbrace{- a_{j_1}}_{\sum_{n=1}^{N-1}(a_{j_{n+1}} - a_{j_n}) - a_{j_N}} + \varepsilon$$

$$\le b_{j_N} + \frac{\varepsilon}{2^{j_N+1}} + \sum_{n=1}^{N-1}\left(b_{j_{n+1}} + \frac{\varepsilon}{2^{j_{n+1}}} - a_{j_n}\right) - a_{j_N} + \varepsilon$$

$$\le \underbrace{\sum_{n=1}^{N}(b_{j_n} - a_{j_n})}_{m_0(I_{j_n})} + \underbrace{\varepsilon + \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n}}_{2\varepsilon}$$

$$\le \sum_{j=1}^{\infty} m_0(I_j) + 2\varepsilon$$