Review Session I:

A) Measures: (Ch. 1)

- algs., σ-alg's, Borel sets
- measures (finite, σ-finite), properties
- premeasure, outer measure, measurable sets, Carathéodory
- Lebesgue (& L–S) measure on $\mathbb{R}$, regularity

Problem 1: $\mu(X) < \infty$, $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$, $\mu(E_j) = \mu(X)$. Show: $\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \mu(X)$

Soln: · look at $\left(\bigcap_{j=1}^{\infty} E_j\right)^c = \bigcup_{j=1}^{\infty} E_j^c$

· $\mu(E_j^c) = \mu(X) - \underbrace{\mu(E_j)}_{\mu(X)} = 0$

· so: $\mu\left(\left(\bigcap_j E_j\right)\right) = \mu(X) - \mu\left(\left(\bigcap_j E_j\right)^c\right) = \mu(X) - \mu\underbrace{\left(\bigcup_j E_j^c\right)}_{\leq \sum_j \mu(E_j^c)}$ sub-add.

$\geq \mu(X)$. $\qquad \sum_j \underbrace{\mu(E_j^c)}_{0}$

$\Rightarrow \mu\left(\bigcap_j E_j\right) = \mu(X)$ ✓

Problem 2: I_1, I_2 disjoint h-intervals, $E_1 \subset I_1$, $E_2 \subset I_2$. Show:

$$m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2)$$

Soln:

- $\leq$: by subadditivity of outer measure
- $\geq$: $\mathcal{A} = \{\perp_\infty$, dis, $\bigcup$'s of h-ints$\}$. Let $\varepsilon > 0$, $\exists$

$$K = \underbrace{\bigcup_{j=1}^{\infty} K_j}_{\in \mathcal{A}} \supset E_1 \cup E_2 \quad \text{s.t.} \quad m^*(E_1 \cup E_2) \geq m(K) - \varepsilon$$

(by def'n of m^*)

$\sum_j m(K_j) \geq - \varepsilon$

- $K \cap I_j \supset E_j$; $m(K) = m(K \cap I_1 \overset{\text{disj't}}{\cup} K \cap I_2) = m(K \cap I_1) + m(K \cap I_2)$ since are measurable

$\Rightarrow m^*(E_1 \cup E_2) \geq \underset{\substack{\vee\!\mid \\ m^*(E_2)}}{m(K \cap I_2)} + \underset{\substack{\vee\!\mid \\ m^*(E_1)}}{m(K \cap I_1)} - \varepsilon$

(monotone) . ε arb $\Rightarrow \geq$. ✓

Integration:

- measurable fns., approx. by simples
- defn. of $\int$ for L^+, for L^1
- convergence thms: MCT, Fatou, DCT

Problem 3: find $f:\mathbb{R}\to\mathbb{R}$ s.t. f is not Lebesgue measurable, but $|f|$ is.

Soln: take $E\subset\mathbb{R}$, $E\notin\mathcal{L}$, $f(x)=\chi_E(x)-\chi_{E^c}(x)$

so $|f|=1$, but $f^{-1}(\{1\})=E\notin\mathcal{L}$. ✓

Problem 4: $f \in L^1(\mu) \cap L^+_1$ find $\displaystyle\lim_{n\to\infty} \int \underbrace{n \log\left(1 + \frac{f(x)}{n}\right)}_{f_n(x)} d\mu(x)$

Soln: · pointwise $f_n(x) \xrightarrow[n\to\infty]{} f(x)$

· and $|f_n(x)| = f_n(x) \leq f(x) \in L^1$.

$\left(\log(1+y) \leq y \text{ since } 1+y \leq e^y \right)$

so DCT $\Rightarrow \displaystyle\lim_{n\to\infty} \int f_n = \int f$ ✓