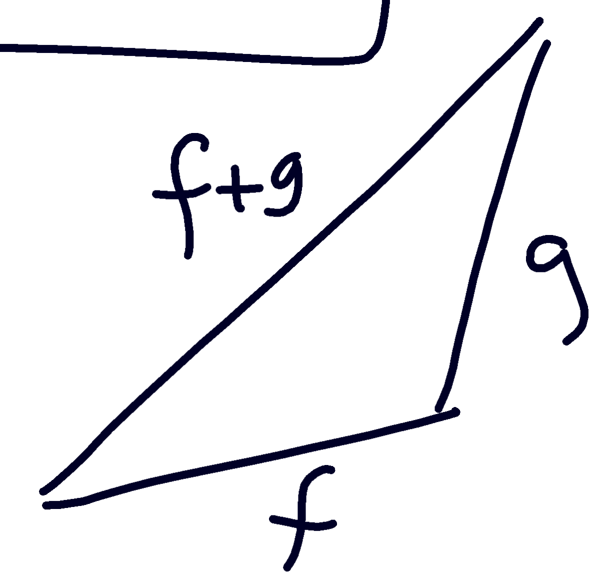
- $\|f\|_p = \left(\int |f|^p \, d\mu\right)^{1/p}$ $\quad L^p = \{ f : X \to \mathbb{C} \mid \|f\|_p < \infty \}$ $\quad 1 \le p < \infty$

- Hölder $\le$: $\left\{ \begin{array}{l} 1 < p < \infty \\ \frac{1}{p} + \frac{1}{q} = 1 \end{array} \right\}$ $\quad \boxed{\|fg\|_1 \le \|f\|_p \|g\|_q}$

<ins>Thm:</ins> (Minkowski $\le$) (i.e. "$\Delta \le$":



$1 \le p < \infty$: $f, g \in L^p$ $\quad \boxed{\|f+g\|_p \le \|f\|_p + \|g\|_p}$

<ins>Pf:</ins>

- $p=1$: $\|f+g\|_p = \int |f+g| \le \int (|f| + |g|) = \|f\|_1 + \|g\|_1$ ✓

$p>1$: $|f+g|^p = |f+g|\,|f+g|^{p-1} \le (|f|+|g|)|f+g|^{p-1} = \underbrace{|f|}_{\in L^p}\underbrace{|f+g|^{p-1}}_{\in L^{\frac{p}{p-1}=q}} + |g|\,|f+g|^{p-1}$

$\left(\frac{1}{p}+\frac{1}{q}=1\right)$

- Hölder (twice) $\Rightarrow$ $\|f+g\|_p^p = \int |f+g|^p$

$$\le \|f\|_p \underbrace{\left\| |f+g|^{p-1} \right\|_{\frac{p}{p-1}}}_{\|f+g\|_p^{p-1}} + \|g\|_p \underbrace{\left\| |f+g|^{p-1} \right\|_{\frac{p}{p-1}}}_{\|f+g\|_p^{p-1}}$$

$$= (\|f\|_p + \|g\|_p)\|f+g\|_p^{p-1}$$

- if $\|f+g\|_p = 0$ ✓, else

$\|f+g\|_p \le \|f\|_p + \|g\|_p$ ✓ $\blacksquare$

goal:
$\|f+g\|_p \le \|f\|_p + \|g\|_p$

Prop: (L^p approx. by simple fns). Given $f \in L^p$, $\varepsilon > 0$, $\exists\, L^p \ni \phi^{\text{simple}}$ s.t. $\|\phi - f\|_p < \varepsilon$.

Pf:

- recall $\exists$ simple $\phi_n \longrightarrow f$ a.e., $|\phi_n| \le |f|$
- thn $\|\phi_n\|_p \le \|f\|_p < \infty \implies \phi_n \in L^p$
- $|\phi_n - f|^p \le (|\phi_n| + |f|)^p \le 2^p |f|^p \in L^1$
- DCT $\implies \|\phi_n - f\|_p^p = \int |\phi_n - f|^p \xrightarrow[n\to\infty]{} 0$ $\blacksquare$

$\underline{\underline{L^\infty}}$: $f : X \to \mathbb{C}$ measurable

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in X} |f| = \inf\{ a \geq 0 \mid \mu(\{x \mid |f(x)| > a\}) = 0 \}$$

$$L^\infty = L^\infty(X, \mathcal{M}, \mu) = \{ f : X \to \mathbb{C} \mid \|f\|_\infty < \infty \}$$

<u>Rem:</u>

- $f \in L^\infty \iff f \underset{a.e.}{=} g$, bounded
- $f \in L^\infty \Rightarrow |f(x)| \leq \|f\|_\infty$ a.e. x

Prop: (properties of L^∞) $\left(\frac{1}{1}+\frac{1}{\infty}=1\right)$

- vector space ✓
- Hölder: $f\in L^1,\ g\in L^\infty \Rightarrow fg\in L^1$, $\|fg\|_1 \le \|f\|_1\|g\|_\infty$

$$\left(|f(x)g(x)| \le \|g\|_\infty |f(x)| \text{ a.e.} \Rightarrow \int |fg| \le \|g\| \int |f| \quad ✓ \right)$$

- Minkowski: $|f(x)+g(x)| \le |f(x)|+|g(x)| \le \|f\|_\infty+\|g\|_\infty$ a.e.

$$\Rightarrow \|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$$

- approx. by simples: $f\in L^\infty$, $\exists\, L^\infty \ni \phi_n^{\text{simple}}$ s.t. $\|\phi_n - f\|_\infty \to 0$

$$\left\{ \begin{array}{l} \cdot\ \phi_n^{\text{simple}} \longrightarrow f \text{ a.e.} \\ \cdot\ |\phi_n - f| \le \frac{1}{2^n} \text{ where } |f| \le 2^n \end{array} \right\} \quad ✓$$