Intro to L^p Spaces (6.1)

$(X, \mathcal{M}, \mu)$ measure space

$f : X \to \mathbb{C}$, measurable

- $0 < p < \infty$

Defn:

- $\|f\|_p := \Big(\int \underbrace{|f|^p}_{\in L^+} \, d\mu \Big)^{1/p} \in [0,\infty]$

- $L^p(X, \mathcal{M}, \mu) = \{ f : X \to \mathbb{C} \text{ meas.}, \ \|f\|_p < \infty \}$

$\Big(L^p(X) = L^p(\mu) = L^p \Big)$

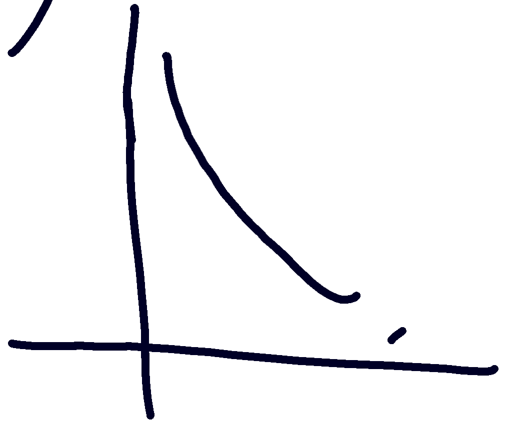
Rems:

- $\|f\|_p = 0 \iff f = 0$ a.e.
- if $\mathcal{L}^p \ni f = g$ a.e., we think of f, g as the same $\in L^p$

Examples:

1. $p = 1$ $\quad L^1(\mu)$, already familiar

2. $L^p(\mathbb{R}, \mathcal{L}, m)$



$x^{-\alpha}\chi_{(0,1)} \in L^p \iff \alpha < \frac{1}{p}$

$\left(\int_0^1 x^{-\alpha p}\,dx < \infty\right)$

$x^{-\alpha}\chi_{(1,\infty)} \in L^p \iff \alpha > \frac{1}{p}$

3. $L^p(\mathbb{R}^n, \mathcal{L}^n, m^n)$

$$\left(\int_0^1 r^{-\alpha p} r^{n-1}\, dr < \infty \right) \longrightarrow |x|^{-\alpha} \chi_{B(1,0)} \in L^p \iff \alpha < \frac{n}{p}$$

$$|x|^{-\alpha} \chi_{B^c(1,0)} \in L^p \iff \alpha > \frac{n}{p}$$

4. $L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \text{counting}) =: \ell^p$

$$\{a_n\}_{n=1}^{\infty} \in \ell^p \iff \|\{a_n\}\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} < \infty$$

Prop: $\boxed{p \geq 1}$ L^p is a vector space

Pf: $\cdot$ $f \in L^p$, $a \in \mathbb{C}$ $\quad \|af\|_p = \left(\int |a|^p |f|^p d\mu\right)^{1/p} = |a| \|f\|_p < \infty$

$\Rightarrow af \in L^p$ ✓

$\cdot$ $f, g \in L^p$ $\quad |f+g|^p \leq \left(2\max(|f|,|g|)\right)^p = 2^p \max(|f|^p, |g|^p)$

$\Rightarrow f+g \in L^p$ $\quad \leq 2^p(|f|^p + |g|^p)$

✓

<u>Thm</u> (Hölder inequality) $1 < p < \infty$,

$1 < q < \infty$ s.t. $\boxed{\frac{1}{p} + \frac{1}{q} = 1}$. If $f \in \mathcal{L}^p$, $g \in \mathcal{L}^q$,

"conjugate" to p

then $fg \in \mathcal{L}^1$ and

$$\boxed{\|fg\|_1 \le \|f\|_p \|g\|_q}$$

<u>Pf:</u>
- can assume $\|f\|_p \neq 0$, $\|g\|_q \neq 0$ $(0 \le 0)$
- " " $f \ge 0$, $g \ge 0$ $(f \mapsto |f|, g \mapsto |g|)$

- by $f \mapsto \frac{f}{\|f\|_p}$, $g \mapsto \frac{g}{\|g\|_q}$, can assume $\|f\|_p = \|g\|_q = 1$

Lemma: $a \geq 0, b \geq 0$, $0 < \lambda < 1$ $\Rightarrow$ $a^{\lambda} b^{1-\lambda} \leq \lambda a + (1-\lambda) b$

(eg. $\lambda = \frac{1}{2}$
$\sqrt{ab} \leq \frac{1}{2}(a+b)$)

(Pf: - $b \neq 0$
- $\underbrace{t^{\lambda} - \lambda t}_{= f(t)} \leq \underbrace{1-\lambda}_{= f(1)}$, $t = \frac{a}{b} \geq 0$
- $\max_{t \geq 0} f(t) = f(1)$, by calculus)

(= $\Leftrightarrow$ $t=1$) ✓

- apply with $a = f^p$
$b = g^q$
$\lambda = \frac{1}{p}$

$$\Rightarrow \underbrace{f^{\overbrace{p/p}^{=1}}\, g^{\overbrace{q(1-p)}^{=1}}}_{fg} \le \frac{1}{p} f^{p} + \underbrace{\left(1-\frac{1}{p}\right)}_{\frac{1}{q}} g^{q}$$

$$\Rightarrow \int fg \le \frac{1}{p}\underbrace{\int f^{p}}_{\|f\|_p^p = 1} + \frac{1}{q}\underbrace{\int g^{q}}_{\|g\|_q^q = 1} = \frac{1}{p} + \frac{1}{q} = 1 \quad \blacksquare$$

<u>Rems.</u>: 1. get $=$ in Hölder $\Leftrightarrow \alpha|f|^p = \beta|g|^q$, some $(\alpha,\beta) \neq (0,0)$

2. classical special case: $p=2$ $(q=2)$, ℓ^2, $N<\infty$, $N=\infty$

$\Rightarrow$ Cauchy-Schwarz: $\displaystyle\sum_{j=1}^{N} |a_j||b_j| \le \sqrt{\sum_{j=1}^{N} |a_j|^2}\sqrt{\sum_{j=1}^{N} |b_j|^2}$ i.e. $|\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$