summary so far: • $F \in NBV \longleftrightarrow m_F$, $\mathbb{C}$ Borel meas. on $\mathbb{R}$

• $F \in NBV \Rightarrow \begin{cases} \bullet\ F' \in L^1(m) \\ \bullet\ m_F \perp m \iff F' = 0 \text{ a.e.} \\ \bullet\ \boxed{m_F \ll m} \iff F(x) = \int_{-\infty}^{x} F'(t)\,dt \quad \text{(FTC)} \end{cases}$

<u>Def</u>: F is AC if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $\{(a_j, b_j)\}_{j=1}^{N}$ (disjoint) s.t. $\sum_{j=1}^{N}(b_j - a_j) < \delta \Rightarrow \sum_{j=1}^{N} |F(b_j) - F(a_j)| < \varepsilon$

<u>Prop</u>: $F \in NBV$. $m_F \ll m \iff F$ is AC

<u>Pf</u>: $\Rightarrow$ ✓

$\Leftarrow$: F AC. Let E be Borel with $m(E)=0$. Goal: $m_F(E)=0$.

Let $\varepsilon>0$. Let δ be as in defn. of AC for F.

- $\exists\ U_1^{\text{open}} \supset E$ s.t. $m(U_1)<\delta$
- $m_F(E) = \inf\{m_F(U) \mid E \subset U^{\text{open}}\}$

$\Rightarrow \exists\ E \subset \cdots \subset U_4 \subset U_3 \subset U_2 \subset U_1$

s.t. $m_F(E) = \lim_{j\to\infty} m_F(U_j) \Leftarrow$

- each $U_j = \bigcup_{k=1}^{\infty}(a_k^j, b_k^j)$ (disjoint), so $\bigcup_{k=1}^{N}(a_k^j, b_k^j) \subset U_j \subset U_1$

$\Rightarrow \sum_{k=1}^{N}(b_k^j - a_k^j) \le m(U_j) \le m(U_1) < \delta \Rightarrow \sum_{k=1}^{N}|\underbrace{F(b_k^j)-F(a_k^j)}_{m_F((a_k^j, b_k^j])}| < \varepsilon$

$\overset{N\to\infty}{\Longrightarrow} |m_F(U_j)| \le \varepsilon \overset{j\to\infty}{\Longrightarrow} |m_F(E)| \le \varepsilon$. So $m_F(E)=0$. $\blacksquare$

Cor:

- $f \in L^1(m) \Rightarrow F(x) = \int_{-\infty}^{x} f(t)\,dt$
  - $\in NBV$ (direct check)
  - is AC ($\mu_F \ll m$)
  - $F' = f$ a.e. (Lebesgue diff. thm.)
- $F \in NBV, AC \Rightarrow F' \in L^1$, $F(x) = \int_{-\infty}^{x} F'(t)\,dt$

($AC \Rightarrow \mu_F \ll m$)

- restrict to interval $[a,b]$, $-\infty < a < b < \infty$

Thm: (FTC for Lebesgue). TFAE

1) $F \in AC([a,b])$
2) $F(x) - F(a) = \int_a^x f(t)\,dt$, some $f \in L^1([a,b], m)$
3) F is a.e diff. (on $[a,b]$), $F' \in L^1([a,b])$, and $F(x) - F(a) = \int_a^x F'(t)\,dt$

$\boxed{1) \Rightarrow 3)}$: ex. $F \in AC[a,b] \Rightarrow F \in BV[a,b]$

$$\tilde{F}(x) = \begin{cases} 0 & x \le a \\ F(x) - F(a) & a \le x \le b \\ F(b) - F(a) & x \ge b \end{cases} \in NBV, \text{ + "Cor"}$$

$\boxed{3) \Rightarrow 2)}$ ✓

$\boxed{2) \Rightarrow 1)}$ extend f by 0 to $\mathbb{R}$ + "Cor" ✓