last time: **Thm** ($\mathbb{C}$ Borel measures on $\mathbb{R}$ $\Longleftrightarrow$ NBV fns)

1. $F \in NBV \Rightarrow \exists!$ $\mathbb{C}$ Borel meas. m_F on $\mathbb{R}$ s.t. $m_F((-\infty,x]) = F(x)$

2. ν $\mathbb{C}$ Borel meas on $\mathbb{R}$ $\Rightarrow$ $F(x) := \nu((-\infty,x]) \in NBV$

Pf: 2. ✓

1. **Lemma:** $F \in BV \Rightarrow$
   - $T_F(-\infty) = 0$
   - F right cont $\Rightarrow T_F$ is right cont.

   Pf: (technical – see text)

   - write $F = F_r + iF_i = \underbrace{F_r^+}_{\frac{1}{2}(F_r + T_{F_r})} - F_r^- + i(F_i^+ - F_i^-)$, $F_{r,i}^{\pm}$ are increasing, right cont., $F_{r,i}^{\pm}(-\infty) = 0$

$\Rightarrow \quad m_F := m_{F_r^+} - m_{F_r^-} + i\,(m_{F_i^+} - m_{F_i^-})$ has desired property

L–S measure

$$m_F((-\infty,x]) = F(x)$$

· uniqueness: each of $\nu^{\pm}_{r,i}$ are determined by values on $(-\infty,x]$ ✓

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note:

$$m_F((a,b]) = F(b)-F(a) = \int_a^b f(t)\,dt, \quad \text{some } f \in L^1$$

$$\iff m_F \ll m$$

· more precisely:

Prop: $F \in NBV$

✓ • $F' \in L^1(m)$ $\iff$ $F(x) = \int_{-\infty}^{x} F'(t)\,dt$ (FTC)

✓ • $m_F \ll m$ $\iff$

✓ • $m_F \perp m$ $\iff$ $F' = 0$ a.e.

Pf:

$$F'(x) = \lim_{h\to 0+} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0+} \frac{\overbrace{m_F((x,x+h])}^{\text{regular}}}{m((x,x+h])} = f(x)$$

$$F'(x) = \lim_{h\to 0+} \frac{F(x-h)-F(x)}{-h} = \lim_{h\to 0+} \frac{m_F((x-h,x])}{m((x-h,x])} = f(x)$$

"shrinking nicely to x"

a.e. x

where $dm_F = \underbrace{d\lambda}_{\perp m} + \underbrace{f\,dm}_{\ll m}$ (L-R-N decomp)

$F' \underset{a.e.}{=} f \in L^1$

$\Rightarrow$

• $m_F \ll m \iff \lambda = 0$

$\Leftarrow$: exercise

• $m_F \perp m \iff F' = f = 0$ ae.

last step: characterize $\mu_F \ll m$ directly in terms of F

<u>Defn:</u> $F : \mathbb{R} \to \mathbb{C}$ is <u>absolutely continuous</u> if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. if $(a_1, b_1), \ldots\ldots, (a_N, b_N)$ are disjoint with $\sum_{j=1}^{N} (b_j - a_j) < \delta$ then $\sum_{j=1}^{N} |F(b_j) - F(a_j)| < \varepsilon$

(and F abs. cont. on $[a,b]$ if this holds for intervals $\subset [a,b]$).

<u>Rem:</u>
- AC $\Rightarrow$ uniform continuity
- F' bounded $\Rightarrow$ F AC

Prop: $F \in NBV$. F is AC $\iff \mu_F \ll m$

Pf: $\Leftarrow$ $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $m(E) < \delta \Rightarrow |\mu_F(E)| < \varepsilon$

Apply this to $E = \bigcup_{j=1}^{N} (a_j, b_j)$.

$\Rightarrow$ next time.