

last time: Thm (\mathbb{C} Borel measures on $\mathbb{R} \iff$ NBV fn's)

1. $F \in \text{NBV} \Rightarrow \exists! \mathbb{C}$ Borel meas. m_F on \mathbb{R}
s.t. $m_F(-\infty, x] = F(x)$

2. ν \mathbb{C} Borel meas on $\mathbb{R} \Rightarrow F(\infty) := \nu(-\infty, \infty) \in \text{NBV}$

Pf: 2. ✓

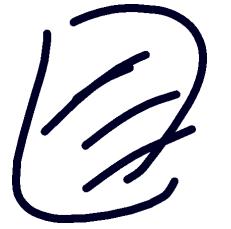
1. Lemma: $F \in \text{BV} \Rightarrow \cdot \bar{T}_F(-\infty) = 0$
 $\cdot F$ right cont $\Rightarrow \bar{T}_F$ is right cont.

Pf: (technical - see text)

• write $F = \bar{F}_r + i\bar{F}_i = \bar{F}_r^+ - \bar{F}_r^- + i(\bar{F}_i^+ - \bar{F}_i^-)$, $\bar{F}_{r,i}^\pm$ are increasing,
right cont., $\bar{F}_{i,i}^\pm(-\infty) = 0$

$\Rightarrow M_F := M_{F_r^+} - M_{F_r^-} + i(M_{F_i^+} - M_{F_i^-})$ has desired property
 L-S measure

uniqueness: each of $N_{r,i}^\pm$ are determined by
 values on $(-\infty, x]$



note: $m_F^{(a,b]} = F(b) - F(a) = \int_a^b f(t) dt$, some $f \in L'$

$$\Leftrightarrow M_F < M$$

* more precisely:

Prop: $F \in NBV$

✓ $F' \in L^1(n)$

✓ $M_F \ll M \Leftrightarrow$

✓ $M_F \perp M \Leftrightarrow F' = 0 \text{ a.e.}$

Pf:

$$F'(x) = \lim_{h \rightarrow 0+} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \rightarrow 0+} \frac{F(x-h) - F(x)}{-h}$$

$$= \lim_{h \rightarrow 0+} \frac{M_F((x, x+h])}{m((x, x+h])}$$

$$= \lim_{h \rightarrow 0+} \frac{M_F((x-h, x])}{m((x-h, x])}$$

$$\begin{aligned} &\xrightarrow{\text{regular}} f(x) \\ &\xrightarrow{\text{"shrink nicely to } x\text{"}} \text{a.e. } x \end{aligned}$$

$\bullet M_F \ll M \Leftrightarrow \lambda = 0$

\Leftrightarrow exercise

$\bullet M_F \perp M \Leftrightarrow F' = 0 \text{ a.e.}$

where

$$dm_F = \underbrace{d\lambda}_{\perp m} + \underbrace{fdm}_{\in m} \quad (\text{L-R-N decomp})$$

$$F' \underset{\text{a.e.}}{=} f \in L'$$

last step: characterize $M_F \ll M$ directly in terms of F

Defn: $F: \mathbb{R} \rightarrow \mathbb{C}$ is absolutely continuous if
 $\forall \varepsilon > 0, \exists S > 0$ s.t. if $(a_1, b_1), \dots, (a_N, b_N)$ are disjoint
with $\sum_{j=1}^N (b_j - a_j) < S$ then $\sum_{j=1}^N |F(b_j) - F(a_j)| < \varepsilon$

(and F abs. cont. on $[a_1, b]$ if this holds for intervals
 $([a_1, b])$.

Rem:

- AC \Rightarrow uniform continuity
- F' bounded $\Rightarrow F$ AC

Prop: $F \in NBV$. \bar{F} is AC $\Leftrightarrow M_F < \infty$

Pf: $\Leftarrow + \varepsilon > 0 \exists \delta > 0$ s.t. $m(E) < \delta \Rightarrow |M_F(E)| < \varepsilon$

Apply this to $E = \bigcup_{j=1}^N (a_j, b_j)$.

\Rightarrow next time.