Properties of Measures (1.3)

definitions: a set X, $\mathcal{M} \subset \mathcal{P}(X)$ a <u>σ-algebra</u> $\left(\begin{array}{l} \cdot\ E \in \mathcal{M} \Rightarrow E^c \in \mathcal{M} \\ \cdot\ \{E_j\}_{j=1}^{\infty} \subset \mathcal{M} \\ \quad \Rightarrow \bigcup_{j=1}^{\infty} E_j \in \mathcal{M} \end{array} \right)$

- a <u>measure</u> on $(X, \mathcal{M})$ is a fn. $\mu : \mathcal{M} \longrightarrow [0, \infty]$ s.t.

  1. $\mu(\emptyset) = 0$
  2. $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ <u>disjoint</u> $\Rightarrow$ $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$

     "countable additivity"

- $(X, \mathcal{M}, \mu)$ is a <u>measure space</u>

  "measurable sets"

Thm: $(X, \mathcal{M}, \mu)$ measure space.

(a) $E, F \in \mathcal{M}$, $E \subset F \Rightarrow \mu(E) \leq \mu(F)$ "monotonicity"

(b) $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M} \Rightarrow \mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j)$ "countable subadditivity"

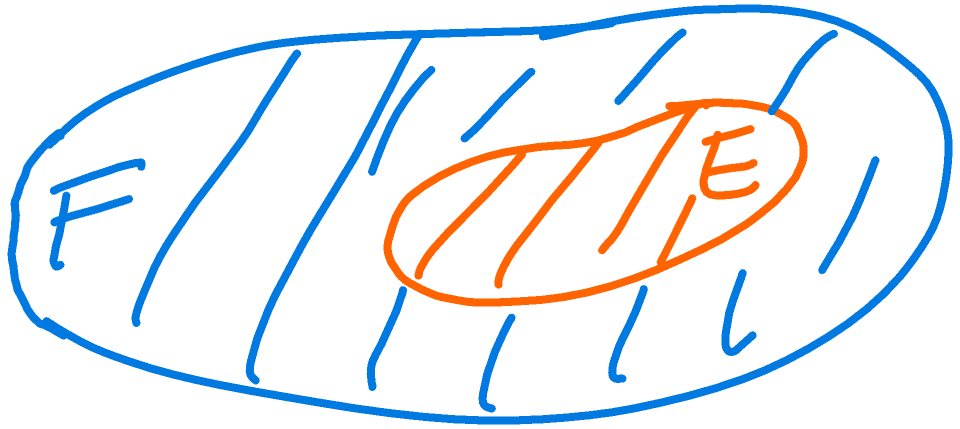
(c) $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$, $E_1 \subset E_2 \subset E_3 \subset \cdots \Rightarrow \mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$

(d) $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$, $E_1 \supset E_2 \supset E_3 \supset \cdots \Rightarrow \mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$

provided $\mu(E_1) < \infty$

"continuity from above/below"

Proof: (a) $F = E \cup (F \setminus E)$ disjoint, $\in \mathcal{M}$ (algebra) $\quad (F \setminus E := F \cap E^c)$



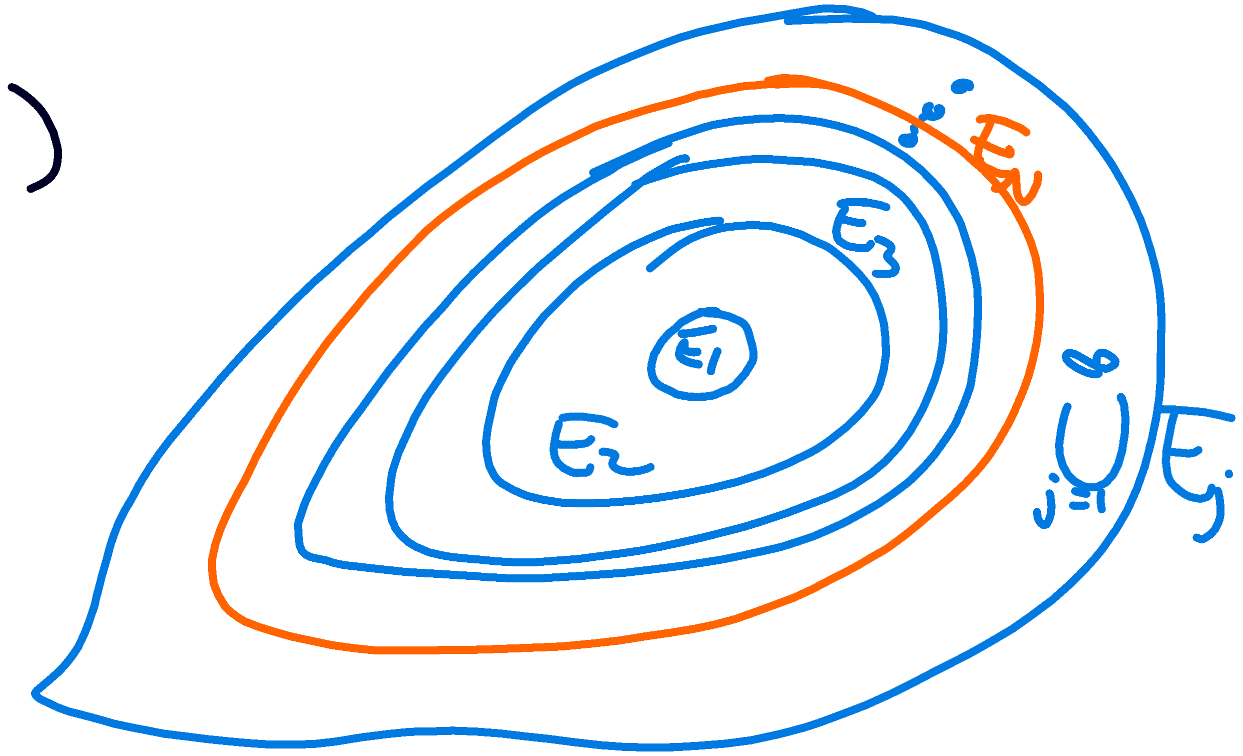
$\in \mathcal{M}$

$\Longrightarrow$ (prop. 2.)

$$\mu(F) = \mu(E) + \underbrace{\mu(F \setminus E)}_{\geq 0} \geq \mu(E) \quad ✓$$

(b) exercise/text

(c)



$$\bigcup_{j=1}^{\infty} E_j = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus E_2) \cup \cdots$$

disjoint, $\in \mathcal{M}$

$\Longrightarrow$ (prop. 2.)

$$\mu\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \sum_{j=1}^{\infty} \mu(E_j \setminus E_{j-1}), \quad E_0 = \emptyset$$

$$= \lim_{N \to \infty} \boxed{\sum_{j=1}^{N} \mu(E_j \setminus E_{j-1})} = \lim_{N \to \infty} \boxed{\mu(E_N)} \quad ✓$$

(d) exercise(text. Why is $\mu(E_1)<\infty$ needed?

→ Ex: $E_j = (j,\infty) \subset \mathbb{R}$, $\bigcap_{j=1}^{\infty} E_j = \emptyset$

μ = counting measure $\mu(E_j)=\infty$, $\mu(\emptyset)=0$

---

more definitions: measure space $(X,\mathcal{M},\mu)$

- is <u>finite</u> if $\mu(X)<\infty$
- is <u>σ-finite</u> if $\exists \{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$, $\mu(E_j)<\infty$ s.t. $X=\bigcup_{j=1}^{\infty} E_j$
- $N \in \mathcal{M}$ is <u>null</u> if $\mu(N)=0$
- measure space is <u>complete</u> if $\mathcal{M}$ includes all subsets of null sets

<u>Rem</u>: any measure sp. can be "completed" in a natural way (Thm. 1.9)

· next: toward Lebesgue measure