$(X, \mathcal{M}) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ $(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$

- goal: for which fns: $F : \mathbb{R} \to \mathbb{C}$

  do we have FTC: $F(x) - F(a) = \int_a^x F'(t)\,dt$

- recall: · if f cont., $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ $(F \in C^1)$ ✓

  1st-year calc.

  · if $f \in L^1_{loc}$ $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ a.e.

  Lebesgue diff.

- last time:
  - defined BV functions
  - $F:\mathbb{R}\to\mathbb{R} \in BV \iff$ F is diff. of 2 increasing, bounded fns.
  - $F \in BV \implies F$ a.e. differentiable

Exercise:

diff. $\forall x$

$x^2\sin\left(\frac{1}{x}\right) \in BV([-1,1])$

$x^2\sin\left(\frac{1}{x^2}\right) \notin BV([-1,1])$

why? recall 1-1 correspondence

(regular) Borel (positive) measures on $\mathbb{R}$ $\longleftrightarrow$ increasing, right cont. fns $F:\mathbb{R}\to\mathbb{R}$ (up to constants)

$$m_F((a,b]) = F(b) - F(a)$$

$$F(x) = \begin{cases} m_F((0,x]) & x>0 \\ 0 & x=0 \\ -m_F((x,0]) & x<0 \end{cases}$$

- pos measures $\to$ $\mathbb{C}$ meas.

$\Rightarrow$ increasing fns $\to$ BV functions

Defn: "normalized" $NBV = \{ F \in BV \mid F$ right cont., $F(-\infty) = 0 \}$

Exercise: $F \in BV \implies F(x+) - F(-\infty) \in NBV$

Thm: 1. $F \in NBV \implies \exists !$ $\mathbb{C}$ Borel measure μ_F s.t. $F(x) = \mu_F((-\infty, x])$

2. μ is a $\mathbb{C}$ Borel measure on $\mathbb{R}$ $\implies F(x) = \mu((-\infty, x]) \in NBV$

<u>Pf</u>: 2. $\nu = \nu_r + i\nu_i \overset{\text{Jordan}}{=} \nu_r^+ - \nu_r^- + i(\nu_i^+ - \nu_i^-)$

(finite, positive) measures

- set $F_{r,i}^{\pm}(x) := \nu_{r,i}^{\pm}((-\infty, x])$
  - increasing ✓
  - right cont. ✓
  - $F(-\infty) = 0$ ✓

$\Rightarrow F = F_r^+ - F_r^- + i(F_i^+ - F_i^-) \in NBV$ ✓

1. more technical $\longrightarrow$ next time