○○○ last time: $F:\mathbb{R}\to\mathbb{R}$ increasing

- F has (at most) countably many discontinuities
- F a.e. diff, $G(x) := F(x+)$, $G' = F'$ a.e.

ex.

- (regular) Borel measures on $\mathbb{R}$ $\longleftrightarrow$ increasing, right cont. fns.

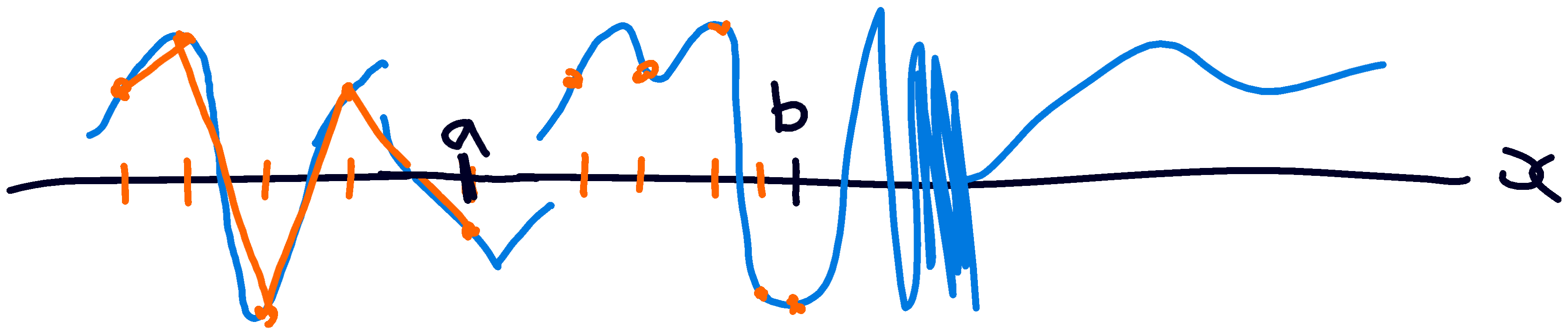
  $m_F((a,b]) = F(b) - F(a)$ $\longleftarrow$ F

- $F(x) = \int_a^x f(t)\,dt$, $f \in L^1_{loc} \cap L^+$ $\Rightarrow$ F incr.

Q: what happens if · measures → complex measures on $\mathbb{R}$?

· $f \in L^1_{loc} \cap L^+ \to f \in L^1_{loc}$?

BV Functions



Defns: $F : \mathbb{R} \to \mathbb{C}$

· the total variation function of F,

$$T_F(x) := \sup \left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \;\middle|\; \begin{array}{l} n \in \mathbb{N} \\ -\infty < x_0 < x_1 < x_2 < \cdots < x_n = x \end{array} \right\}$$

- if $a < b$,

$$T_F(b) = T_F(a) + \underbrace{\sup\left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \;\middle|\; \begin{array}{c} n \in \mathbb{N} \\ a = x_0 < x_1 \\ < \dots < x_n = b \end{array} \right\}}_{\text{total variation of } F \text{ on } [a,b]}$$

- F is of <u>bounded variation</u>, $F \in BV$ if $T_F(\infty) < \infty$
- $F \in BV([a,b])$ if the tot. var. of F on $[a,b]$ is $< \infty$

Examples:

1. $F:\mathbb{R}\to\mathbb{R}$ increasing $\in BV \iff F$ is bounded

$$\left(\sum_{j=1}^{n} |F(x_j)-F(x_{j-1})| = \sum_{j=1}^{n} \big(F(x_j)-F(x_{j-1})\big) = F(x_n)-F(x_0) \right)$$

2. $F:\mathbb{R}\to\mathbb{C}$ differentiable, with F' bounded: $|F'(x)|\le M$

$$\Rightarrow |F(x_j)-F(x_{j-1})| \underset{MVT}{=} |F'(x_j^*)(x_j-x_{j-1})| \le M(x_j-x_{j-1})$$

$$\Rightarrow \sum_{j=1}^{n} |F(x_j)-F(x_{j-1})| \le M(x_n-x_0) \quad \Rightarrow F\in BV([a,b])$$

$$-\infty<a<b<\infty$$

3. BV is linear: $F, G \in BV$, $a, b \in \mathbb{C}$

$\Rightarrow aF + bG \in BV$ (exercise)

<u>Lemma:</u> $F : \mathbb{R} \to \mathbb{R} \in BV \Rightarrow T_F \pm F$ are increasing

<u>Corollary:</u> "Jordan decomposition" for BV fns:

$$F = \underbrace{\tfrac{1}{2}(T_F + F)}_{\text{"pos. var. of } F\text{"}} - \underbrace{\tfrac{1}{2}(T_F - F)}_{\text{"neg. var. of } F\text{"}}$$

$\therefore$ can apply, for $F : \mathbb{R} \to \mathbb{C} \in BV$,

to $\underbrace{\mathrm{Re} F}_{\in BV}$ and $\underbrace{\mathrm{Im} F}_{\in BV}$.

$\Rightarrow \begin{cases} \cdot\ F(x-),\ F(x+) \text{ exist } \forall x \\ \cdot \text{ set of discontinuities is (at most) countable} \\ \cdot\ F \text{ is diff. a.e., and } G(x) := F(x+) \Rightarrow G' = F' \text{ a.e.} \end{cases}$

Pf of lemma: let $x<y$, $\varepsilon>0$

$\exists\ -\infty < x_0 < x_1 < \dots < x_n = x$

s.t. $\displaystyle\sum_{j=1}^{n} |F(x_j)-F(x_{j-1})| \geq T_F(x)-\varepsilon$

- $\displaystyle T_F(y) \geq |F(y)-F(x)| + \sum_{j=1}^{n} |F(x_j)-F(x_{j-1})| \geq \underbrace{|F(y)-F(x)|}_{\geq \pm(F(y)-F(x))} + T_F(x)-\varepsilon$

$\Rightarrow T_F(y) \mp F(y) \geq \pm(F(y)-F(x)) \mp F(y) + T_F(x) - \varepsilon$

$= T_F(x) \mp F(x) - \varepsilon$

ε arbitrary

$\Longrightarrow\ T_F(y) \mp F(y) \geq T_F(x) \mp F(x)$. $\blacksquare$