Differentiation on $\mathbb{R}$ (3.5) $\quad (\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$

- know: $f \in L^1_{loc}$, $F(x) = \int_a^x f(t)\,dt$

  $\Rightarrow$ F is diff. a.e. and $F' = f$ a.e.

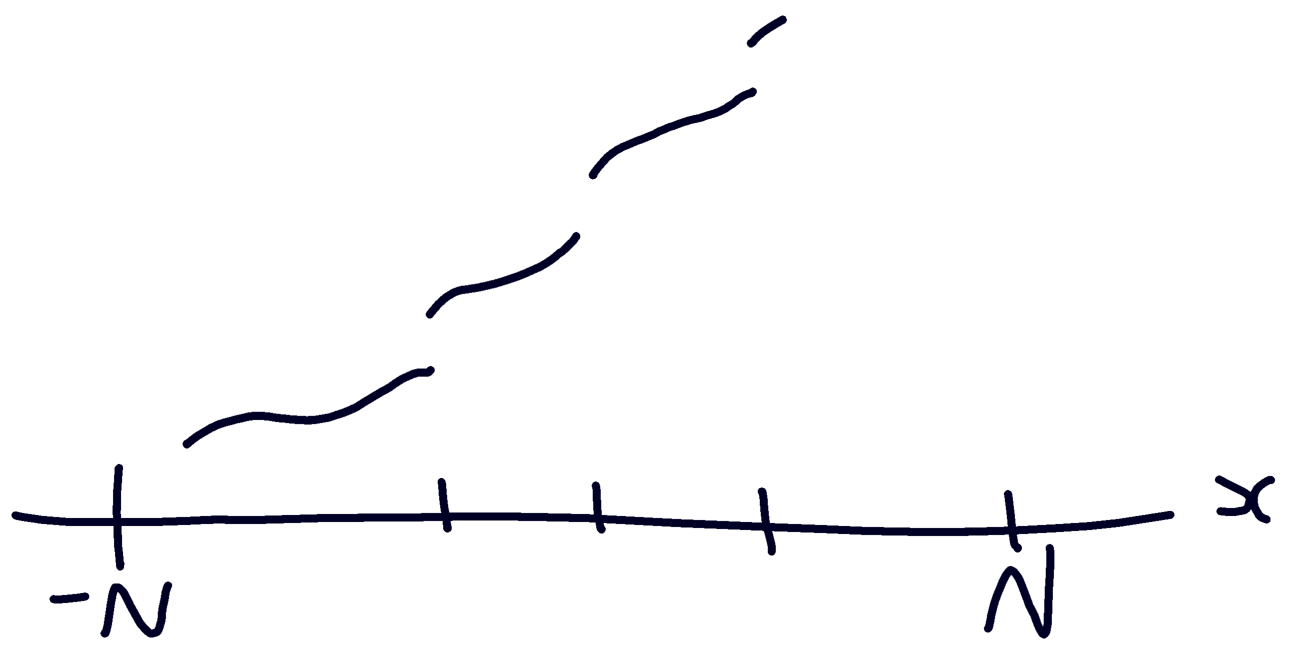
- goal: characterize such F

  (i.e. for which F do we have a FTC?)

Increasing functions: $x < y \Rightarrow F(x) \le F(y)$

Thm: $F: \mathbb{R} \to \mathbb{R}$ increasing

1. F has countably many discontinuities ✓
2. F is differentiable a.e.
3. $G(x) := F(x+)$, then G is diff. a.e, and $G' = F'$ a.e.



Pf: 1. consider disjoint intervals $\{(F(x-), F(x+))\}_{x \in \mathbb{R}}$
(since increasing) $(x < y \Rightarrow F(x+) \le F(y))$

So $|x| \le N \Rightarrow F(-N) \le F(x-) \le F(x+) \le F(N)$

$$\Rightarrow \bigcup_{|x| \le N} (F(x-), F(x+)) \subset [F(-N), F(N)]$$

$$\Rightarrow \sum_{|x| \le N} (F(x+) - F(x-)) \le F(N) - F(-N) < \infty$$

$\Rightarrow F(x+) > F(x-)$ for countably many x, $|x| \le N$ ✓

3. G is increasing, right continuous

$\hookrightarrow$ L-S measure m_G, regular

s.t. $$G(x) = \begin{cases} m_G((0,x]) & x>0 \\ 0 & x=0 \\ -m_G((x,0]) & x>0 \end{cases}$$

so $$\frac{G(x+h)-G(x)}{h} = \frac{1}{h}\begin{cases} m_G(x,x+h] & h>0 \\ -m_G(x+h,h] & h<0 \end{cases}$$

lebesgue decomp $$m_G = \underset{\perp m}{\lambda} + \underset{\ll m}{\rho}, \quad d\rho = \frac{d\rho}{dm}\,dm$$

- so as $h \to 0+$ $\quad (x, x+h]$ shrinks "nicely" to x:

$$\begin{cases} \cdot\ (x, x+h] \subset (x-2h, x+2h) \\ \cdot\ m(x, x+h] \geq \frac{1}{4} m((x-2h, x+2h)) \end{cases}$$

$$\Rightarrow \frac{G(x+h) - G(x)}{h}$$

$$= \frac{m_G((x, x+h])}{m((x, x+h])} \longrightarrow \frac{d\rho}{dm}(x) \ \underbrace{\text{a.e.}}_{m} \ x$$

- also, as $h \to 0-$, $\quad \dfrac{G(x+h) - G(x)}{h} \longrightarrow \dfrac{d\rho}{dm}$ a.e. x

$\Rightarrow$ G is a.e. differentiable

2. $H = \underset{F(x+)}{\underset{\|}{G}} - F \geq 0$

- $\underbrace{\underbrace{\{H \neq 0\}}_{\text{countable}} = \{x_1, x_2, \ldots\}}$

$\underbrace{|x| \leq N \qquad \sum_{j,\, |x_j| \leq N} H(x_j) < \infty}_{}$ (as above)

regular

- set $\mu = \sum_j H(x_j) \delta_{x_j}$, regular

- $\mu \perp m \Rightarrow \left|\frac{H(x+h) - H(x)}{h}\right| \leq \frac{H(x+h) + H(x)}{h} \leq \frac{2\mu([x, x+h])}{h} \to 0$ a.e.

$\Rightarrow H' = 0$ a.e. x

$\Rightarrow F$ is diff. a.e., $F' = G'$ a.e.