recall: $f \in L^1_{loc}$, $\qquad (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, m)$

$x \in L_f \Rightarrow \dfrac{1}{m(E_r)} \displaystyle\int_{E_r} |f(y) - f(x)|\,dy \xrightarrow[r \to 0]{} 0$ if E_r shrinks nicely to x $(r \to 0+)$

& $m(L_f^c) = 0$

so in particular $\dfrac{1}{m(E_r)} \displaystyle\int_{E_r} f \xrightarrow{r \to 0+} f(x)$ a.e. x

- for $n=1$, for $f \in L^1_{loc}$, set

$$F(x) = \int_a^x f(t)\,dt. \quad \text{Then}$$

$$\lim_{h \to 0+} \frac{F(x+h)-F(x)}{h} = \lim_{h \to 0+} \frac{1}{h} \int_{\underbrace{(x,x+h)}_{\text{shrinks nicely to } x \text{ as } h \to 0+}} f(t)\,dt = f(x) \quad \text{a.e. } x$$

- same with $\lim_{h \to 0-}$

So: $F(x)$ is differentiable a.e., with $F' = f$ a.e. x

(generalization of FTC)

differentiation of measures:

$$A_r f(x) = \frac{\nu(B(r,x))}{m(B(r,x))}$$

when $d\nu = f\,dm$

<u>Def:</u> a measure ν on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ is <u>regular</u> if

$$\begin{cases} 1.\ \nu(K) < \infty \text{ for } K \text{ compact} \\ 2.\ \nu(E) = \inf\{\nu(U) \mid U \overset{\text{open}}{\supset} E\} \ \forall E \in \mathcal{B}_{\mathbb{R}^n} \end{cases}$$

(Rem: in fact 1. $\Rightarrow$ 2.)

- ν signed or complex is regular if $|\nu|$ is

Examples: (exercise/text)

1. $f \in L^+$, $d\nu = f\,dm$ is regular $\iff$ $f \in L^1_{loc}$

2. $F : \mathbb{R} \to \mathbb{R}$ increasing, right-continuous, L-S m_F is regular

Thm: ν a regular (signed or complex) Borel measure on $\mathbb{R}^n$, let $d\nu = \underbrace{d\lambda}_{\perp m} + \underbrace{f\,dm}_{\ll m}$ be its L-R-N decomp (w.r.t. m)

Then for m - a.e. x,

$$\frac{\nu(E_r)}{m(E_r)} \xrightarrow[r \to 0+]{} f(x)$$

$\forall \ \{E_r\}_{r>0}$ shrinking nicely to x

Rem: regular $\Rightarrow$ σ-finite

Pf: • ν regular $\overset{\text{(exercise)}}{\Rightarrow}$ λ regular and $f \in L^1_{loc}$

$\Rightarrow$ suffices to show $\dfrac{\lambda(E_r)}{m(E_r)} \xrightarrow[r \to 0+]{} 0$ m a.e. x

- $\left|\frac{\lambda(E_r)}{m(E_r)}\right| \le \frac{|\lambda|(E_r)}{m(E_r)} \le \frac{|\lambda|(B(r,x))}{m(E_r)} \le \frac{1}{\alpha}\underbrace{\left[\frac{|\lambda|(B(r,x))}{m(B(r,x))}\right]}$
- $E_r \subset B(r,x)$
- $m(E_r) \ge \alpha\, m(B(r,x))$

$\Rightarrow$ can assume

- $\lambda \ge 0$
- $E_r = B(r,x)$

- $\lambda \perp m \Rightarrow \lambda(A) = m(A^c) = 0$

can ignore

$$F_k := \left\{ x \in A \;\middle|\; \overline{\lim_{r \to 0}} \frac{\lambda(B(r,x))}{m(B(r,x))} > \frac{1}{k} \right\} \quad , k = 1, 2, 3, \ldots$$

- suffices to show $m(F_k) = 0$ $\left(\text{then } m\left(\bigcup_{k=1}^{\infty} F_k \right) = 0 \right)$

- λ regular $\Rightarrow \forall \varepsilon > 0$, $A \subset U^{\text{open}}$, s.t. $\lambda(U) < \varepsilon$

- if $x \in F_k \subset A \subset U \Rightarrow$ · $\exists$ $B(r,x) \subset U$

  - and some ball $B_x \subset U$ s.t. $\lambda(B_x) > \frac{1}{k} m(B_x)$

• if $c < m(F_k) \leq m\left(\bigcup_{x \in F_k} B_x\right)$

$\left(F_k \subset \bigcup_{x \in F_k} B_x\right)$ so covering lemma

$\Rightarrow \exists\ B_{x_1}, \dots B_{x_\ell}$, disjoint,

s.t. $\sum_{j=1}^{\ell} m(B_{x_j}) > \frac{c}{3^n}$

$$\Rightarrow c < 3^n \sum_{j=1}^{\ell} \underbrace{m(B_{x_j})}_{< k\lambda(B_{x_j})} < 3^n k \sum_{j=1}^{\ell} \lambda(B_{x_j}) \leq 3^n k \lambda(U) < 3^n k \varepsilon$$

• ε arbitrary $\Rightarrow m(F_k) = 0$ $\blacksquare$