ooo leftover:

<u>Covering Lemma</u>: $\mathcal{C}$ any collection of (open) balls in $\mathbb{R}^n$.

If $c < m\left(\bigcup_{B\in\mathcal{C}} B\right)$, $\exists$ disjoint $B_1, \dots, B_k \in \mathcal{C}$ s.t.

$$\sum_{j=1}^{k} m(B_k) > \frac{c}{3^n}.$$

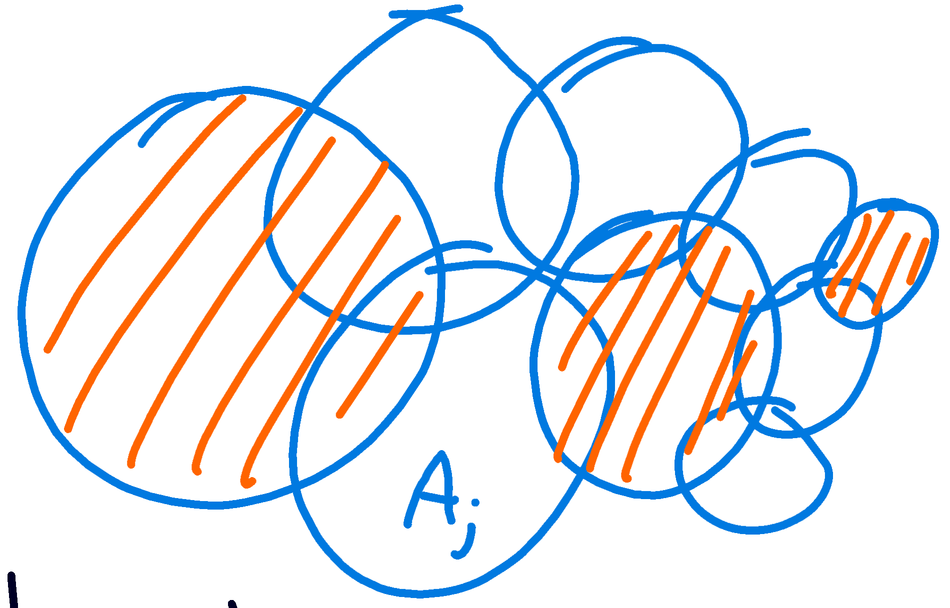
<u>Pf:</u> $\cdot$ $U := \bigcup_{B\in\mathcal{C}} B$, $c < m(U)$

$= \sup\{m(K) \mid K \overset{\text{cmp}}{\subset} U\}$

- $\exists\ K \overset{\text{compact}}{\subset} U$ s.t. $m(K) > c$
- K compact $\Rightarrow \exists$ balls $A_1, A_2, \dots, A_m \in \mathcal{C}$ s.t. $K \subset A_1 \cup A_2 \cup \dots \cup A_m$
- B_1 = largest of A_j's
- B_2 = " " " not intersecting B_1

∘ ∘ ∘ etc., until choices are exhausted

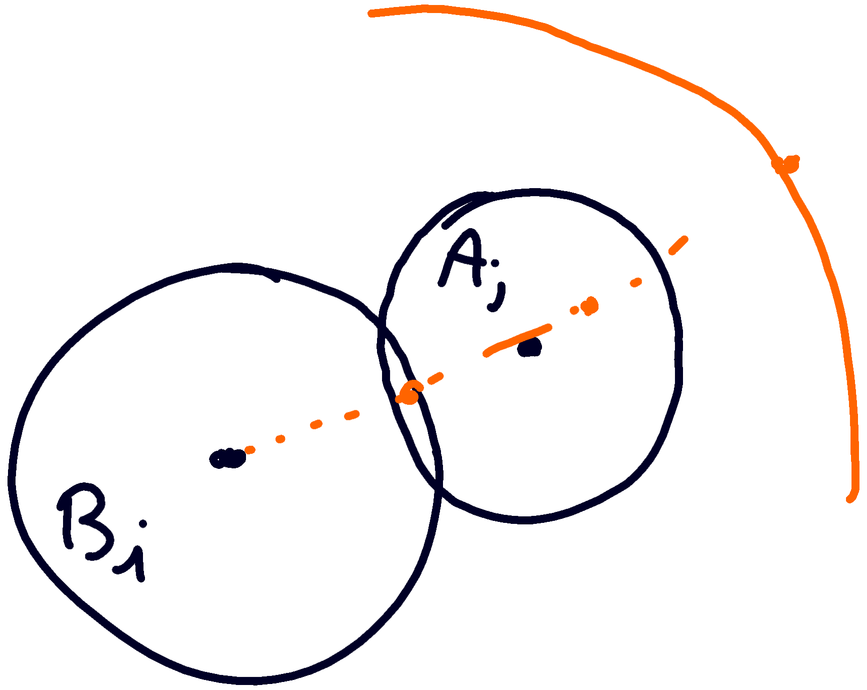
$\longrightarrow B_1, B_2, \dots, B_k$



- if $A_j \not\subset \{B_1, \dots, B_k\}$, then $A_j \cap B_i \neq \phi$, some i;

  & if B_i is the 1st such, $\text{radius}(A_j) \leq \text{radius}(B_i)$

  - set $B_i^* := B_i$ with radius tripled



$$\Rightarrow \quad A_j \subset B_i^*$$

- so $K \subset \bigcup_{j=1}^{m} A_j \subset B_1^* \cup \cdots B_k^*$

$$\Rightarrow \quad c < m(K) \leq \sum_{j=1}^{k} \underbrace{m(B_j^*)}_{3^n m(B_j)} = 3^n \sum_{j=1}^{k} m(B_j)$$

$\blacksquare$

recall, we proved: $f \in L^1_{loc}$ $\quad A_r f(x) \xrightarrow[r \to 0]{} f(x)$ for a.e. x

i.e. $\displaystyle \frac{1}{m(B(r,x))} \int_{B(r,x)} [f(y) - f(x)]\, dy \xrightarrow[r \to 0]{} 0$, a.e. $x \in \mathbb{R}^n$

goal: mild strengthenings of this. The Lebesgue set of f is

$$L_f := \left\{ x \;\middle|\; \lim_{r \to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - f(x)|\, dy = 0 \right\}$$

Thm: $f \in L^1_{loc}$, $m(L_f^c) = 0$

Pf: • for any $z \in \mathbb{C}$, since $|f - z| \in L^1_{loc}$

$$\Longrightarrow \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - z|\, dy \xrightarrow[r \to 0]{} |f(x) - z|$$

$$\forall\ x \notin E_z, \quad m(E_z) = 0$$

- let D be a countable, dense, subset of $\mathbb{C}$

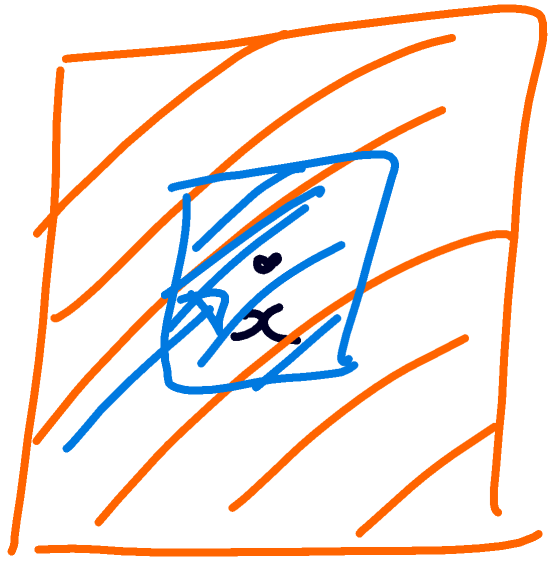
$\therefore$ set $E = \bigcup_{z \in D} E_z$, so $m(E) = 0$

- for $x \notin E$, $\forall \varepsilon > 0$ $\exists$ $z \in D$ s.t. $|f(x) - z| < \varepsilon$, so

$$\overline{\lim_{r \to 0}} \frac{1}{|B(r,x)|} \int_{B(r,x)} \underbrace{|f(y) - f(x)|}_{\leq |f(y) - z| + \underbrace{|f(x) - z|}_{< \varepsilon}} \leq |f(x) - z| + \varepsilon < 2\varepsilon.$$

Since ε arbitrary, $\lim_{r \to 0} = 0$, $x \in L_f$. $\blacksquare$

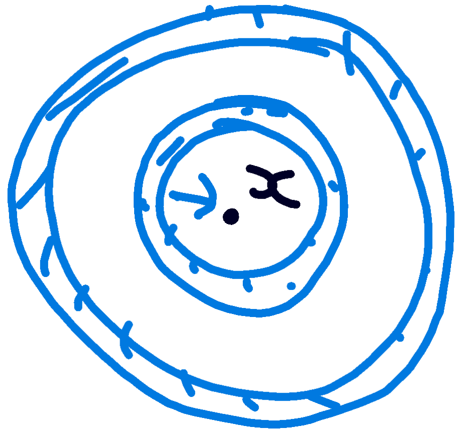
2. <u>Thm</u> (Lebesgue differentiation theorem):



If $f \in L^1_{loc}$, $x \in L_f$. For any family $\{E_r\}_{r>0}$ of Borel sets s.t.

1. $E_r \subset B(r,x) \quad \forall r$
2. $\exists\, \alpha > 0$ s.t. $m(E_r) \geq \alpha\, m(B(r,x)) \quad \forall r$

E_r "shrinks nicely to x"



$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)|\, dy = 0.$$

<u>Pf</u>:

$$\frac{1}{m(E_r)}\int_{E_r\subset B(r,x)}|f(y)-f(x)|\,dy \le \frac{1}{m(E_r)}\int_{B(r,x)}|f(y)-f(x)|\,dy$$

$$\le \frac{1}{\alpha}\,\frac{1}{m(B(x,r))}\int_{B(r,x)}|f(y)-f(x)|\,dy \xrightarrow[r\to 0]{} 0$$

since $x\in L_f$. $\blacksquare$

3. extend to measures:

next time ...