last time: $(X, \mathcal{M}, \nu) = (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, m^n = m)$

- $f \in L^1_{loc}$, $x \in \mathbb{R}^n$, $r > 0$: $A_r f(x) = \fint_{B(r,x)} f = \frac{1}{m(B(r,x))} \int_{B(r,x)} f$

- Q: $A_r f(x) \xrightarrow{r \to 0} f(x)$ a.e. x ?

(a "FTC")

Def: for $f \in L^1_{loc}$, its **Hardy-Littlewood Maximal Function** is

$$Hf(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f| = \sup_{r>0} A_r|f|(x)$$

Thm (maximal theorem). $\exists\, C = C(n) > 0$ s.t.

if $f \in L^1$, and $\alpha > 0$

$$m\left(\{x \mid Hf(x) > \alpha\}\right) \le \frac{C}{\alpha} \int |f|.$$

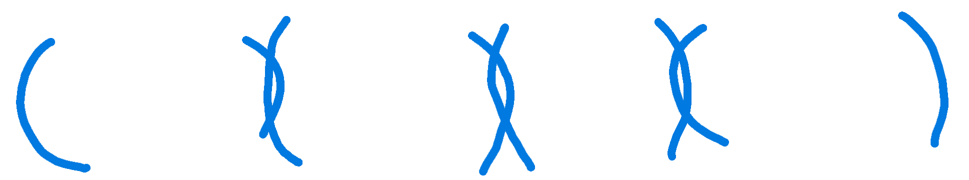
Pf: • $E_\alpha := \{ Hf(x) > \alpha \}$

• if $x \in E_\alpha$, $\exists \; r_x > 0$ s.t. $\underbrace{A_{r_x}|f|(x)}_{\frac{1}{m(B_{r_x,x})}\int_{B(r_x,x)}|f|} > \alpha$

• $E_\alpha \subset \bigcup_{x \in E_\alpha} B(r_x, x)$

Lemma (Covering): if $\mathcal{C}$ is a collection of balls in $\mathbb{R}^n$, and $c < m\left(\bigcup_{B \in \mathcal{C}} B\right)$, then $\exists$ disjoint $B_1, B_2, \dots, B_k \in \mathcal{C}$

s.t. $\sum_{j=1}^{k} m(B_j) > \frac{c}{3^n}$



so: if $\quad c < m(E_\alpha) \leq m\left(\bigcup_{x \in E_\alpha} B(r_x, x)\right)$,

$\exists \quad B(r_{x_1}, x_1), \ldots, B(r_{x_k}, x_k)$, disjoint, s.t.

$$\sum_{j=1}^{k} m(B(r_{x_j}, x_j)) > \frac{c}{3^n}$$

$$\Rightarrow \quad c < 3^n \sum_{j=1}^{k} \underbrace{m(B(r_{x_j}, x_j))}_{\leq \frac{1}{\alpha} \int_{B(r_{x_j}, x_j)} |f|} \leq \frac{3^n}{\alpha} \int |f|$$

$$\Rightarrow m(E_\alpha) \leq \frac{3^n}{\alpha} \int |f| \qquad \blacksquare$$

<u>Thm:</u> if L^1_{loc}, then $A_r f(x) \xrightarrow[r \to 0]{} f(x)$ a.e. $x \in \mathbb{R}^n$

<u>Pf:</u> · suffices to prove this for a.e. x in $B(0,R)$, any $R>0$

· if $x \in B(0,R)$, $r \le 1$, then $A_r f(x) = A_r \underbrace{\chi_{B(0,R+1)} f}_{L^1}(x)$

$\Rightarrow$ can assume $f \in L^1$

<u>1</u>. check it for continuous $f \in L^1$ (classical FTC)

· for $x \in \mathbb{R}^n$, for any $\delta > 0$, $\exists\ r > 0$ s.t.

$$|y-x| < r \Rightarrow |f(y) - f(x)| < \delta$$

$$\Rightarrow \left|A_r f(x) - f(x)\right| = \frac{1}{B(r,x)} \left| \int_{B(r,x)} [f(y) - f(x)]\, dy \right|$$

$$\leq \frac{1}{B(r,x)} \int_{B(r,x)} \delta \leq \delta$$

i.e. $$A_r f(x) \xrightarrow[r \to 0]{} f(x) \quad \checkmark$$

2. if $f \in L^1$, $\varepsilon > 0$, $\exists$ continuous $\underline{g \in L^1}$

s.t. $$\int |f - g| < \varepsilon$$

then: $$|A_r f(x) - f(x)| \le \underbrace{|A_r(f-g)(x)|}_{\substack{\le A_r|f-g|(x) \\ \le H(f-g)(x)}} + |A_r g(x) - g(x)| + |g(x) - f(x)|$$

$$\Rightarrow \overline{\lim_{r\to 0}} |A_r f(x) - f(x)| \le H(f-g)(x) + 0 + |f(x) - g(x)|$$

- for $\alpha > 0$, let $E_\alpha = \{ x \mid \overline{\lim_{r\to 0}} |A_r f(x) - f(x)| > \alpha \}$

$$\Rightarrow m(E_\alpha) \le m(\{H(f-g) > \alpha/2\}) + m(\{|f-g| > \alpha/2\})$$

- maximal thm $\Rightarrow$ $m\left(\left\{H(f-g) > \frac{\alpha}{2}\right\}\right) \leq \frac{2C}{\alpha} \underbrace{\int |f-g|}_{< \varepsilon}$

- $m\left\{|f-g| > \frac{\alpha}{2}\right\} \cdot \frac{\alpha}{2} \leq \int_{\{|f-g| > \alpha/2\}} |f-g| \leq \int |f-g| < \varepsilon$

$$\Rightarrow m(E_\alpha) \leq \frac{2C\varepsilon}{\alpha} + \frac{2\varepsilon}{\alpha}$$

$$\Rightarrow m(E_\alpha) = 0$$

- $A_r f(x) \to f(x) \quad \forall\, x \notin \bigcup_{n=1}^{\infty} E_{1/n}$, null. $\blacksquare$