∘∘∘ previously: $\nu, \mu \geq 0$ (σ-finite) on $(X, \mathcal{M})$

- Lebesgue decomp: $\nu = \underset{\perp \mu}{\lambda} + \underset{\ll \mu}{\rho}$ (unique)

- R-N theorem: $\rho \ll \mu \Rightarrow d\rho = f\, d\mu$ ($f : X \to \mathbb{R}$ unique μ-a.e.)

<u>Rems:</u> 1. the R-N derivative $\boxed{f = \frac{d\rho}{d\mu}}$ obeys familiar rules, eg:

- $$\underbrace{\frac{d(\nu_1 + \nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}}_{\mu\text{ - a.e.}}$$

(precisely: if $\nu_1, \nu_2 \ll \mu$ $\Rightarrow \nu_1 + \nu_2 \ll \mu$, and formula holds

- $\int g\,d\nu = \int g \frac{d\nu}{d\mu}\,d\mu$ (precisely: if $g \in L^1(\nu)$ and $\nu \ll \mu$, then $g\frac{d\nu}{d\mu} \in L^1(\mu)$ and formula holds)
- $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$ (i.e: $\nu \ll \mu \ll \lambda \Rightarrow \nu \ll \lambda$ and formula holds λ-a.e.)

② The L-R-N theory extends naturally to <u>complex measures</u>: $\nu : \mathcal{M} \longrightarrow \mathbb{C}$ (no ∞ allowed)

- $\nu(\emptyset) = 0$
- $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ disjoint $\Rightarrow$ $\nu(\bigcup_j E_j) = \underbrace{\sum_j \nu(E_j)}_{\text{abs. convergent}}$

via $\nu = \nu_r + i\nu_i$

Read: 3.3

Differentiation on $\mathbb{R}^n$ (3.4) $(X, \mathcal{M}) = (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$

- classical FTC: $f : \mathbb{R} \to \mathbb{R}$ continuous

$$f(x) = \frac{d}{dx}\int_a^x f(t)\,\underset{\substack{\uparrow \\ \text{Lebesgue measure}}}{dt} = \lim_{r \to 0} \frac{1}{r}\int_x^{x+r} f(t)\,dt$$

$$\underset{(\text{if } \lim \exists)}{=} \lim_{r \to 0+} \underbrace{\frac{1}{2r}\int_{x-r}^{x+r} f(t)\,dt}_{\substack{\text{avg. value of} \\ f \text{ over "ball" } (x-r,\, x+r)}}$$

- goals: generalize to
  - $\mathbb{R}^n$
  - "wilder" f (eg. discontinuous)
  - more general sets than "balls"
  - measures:

eg: $d\nu = f\,dm$, then

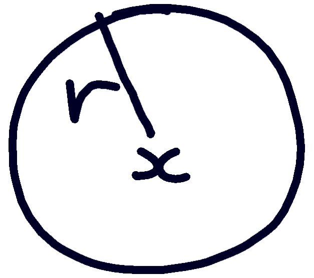
$$\lim_{r \to 0+} \frac{\nu((x-r, x+r))}{m((x-r, x+r))} \stackrel{?}{=} f(x) \qquad \left(f = \frac{d\nu}{dm}\right)$$

<u>Defn:</u> measurable $f : \mathbb{R}^n \to \mathbb{C}$ is <u>locally integrable</u>, written $f \in L^1_{loc}$ if $\int_K |f| < \infty$ $\forall$ bounded, $\underbrace{\text{measurable}}_{\text{(Borel)}}$ K

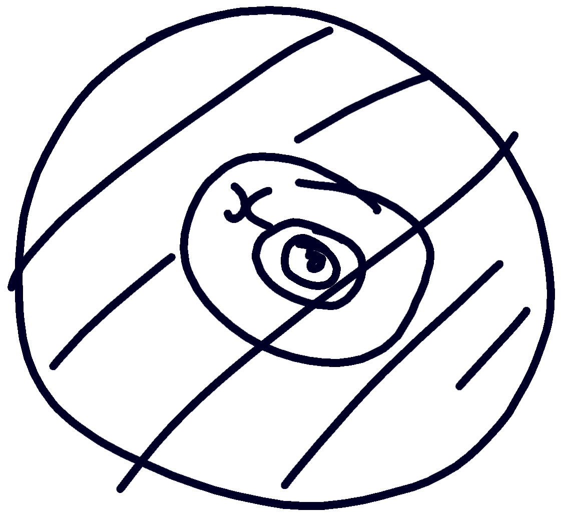
(Lebesgue measure)

· for $f \in L^1_{loc}$, $r>0$, $x \in \mathbb{R}^n$, define

$$A_r f(x) = \frac{1}{m(B(r,x))} \int_{B(r,x)} f = \fint_{B(r,x)} f$$



$B(r,x)$ (open) balls



Q: is $\lim_{r \to 0+} A_r f(x) = f(x)$?

(a.e. x)

Lemma: $A_r f(x)$ is continuous in (r, x) $\quad r > 0$, $x \in \mathbb{R}^n$

Pf: • $A_r f(x) = \underbrace{\frac{1}{c r^n}}_{\text{cont.}} \int_{B(r,x)} f$, so consider just $\underbrace{\int_{B(r,x)} f}$

$$\int \chi_{B(r,x)} f$$

• if $(r, x) \to (r_0, x_0)$, $\quad r_0 > 0$, $x_0 \in \mathbb{R}^n$

$$\chi_{B(r,x)}(y) \longrightarrow \chi_{B(r_0,x_0)}(y)$$

$\forall y \notin \partial B(r_0, x_0)$

boundary $\quad$ in particular <u>a.e.</u>

so: • $f(y)\chi_{B(r,x)}(y) \longrightarrow f(y)\chi_{B(r_0,x_0)}$ a.e.

• $\left|f(y)\chi_{B(r,x)}(y)\right| \le |f(y)|\chi_{\underset{\sim\sim\sim}{B(r_0+1,x_0)}}(y) \in L^1$

for (r,x) close to (r_0,x_1)

• DCT $\Longrightarrow \int_{B(r,x)} f \longrightarrow \int_{B(r_0,x_0)} f.$ ✓