ooo so far: $\cdot$ signed measures $\nu : \mathcal{M} \to (-\infty, \infty]$ or $[-\infty, \infty)$

$\longrightarrow$ $\cdot$ ν, μ "mutually singular", $\nu \perp \mu$ if $\underset{\text{disjoint}}{X = A \cup B}$, B μ-null, A ν-null

$\longrightarrow$ $\cdot$ Hahn decomp: $\underset{\text{disjoint}}{X = P \cup N}$, P ν-positive, N ν-negative (unique up to null sets)

$\longrightarrow$ $\cdot$ Jordan decomp: $\nu = \nu^+ - \nu^-$, $\nu^+ \geq 0$, $\nu^- \geq 0$ (unique)

$\left(\begin{array}{l} \nu^+ = \nu\lceil_P \\ \nu^- = -\nu\lceil_N \end{array}\right)$ $\longrightarrow$ $\cdot$ $|\nu| := \nu^+ + \nu^-$ the total variation measure of ν

Exercise: $\cdot$ E ν-null $\iff$ $|\nu|(E) = 0$

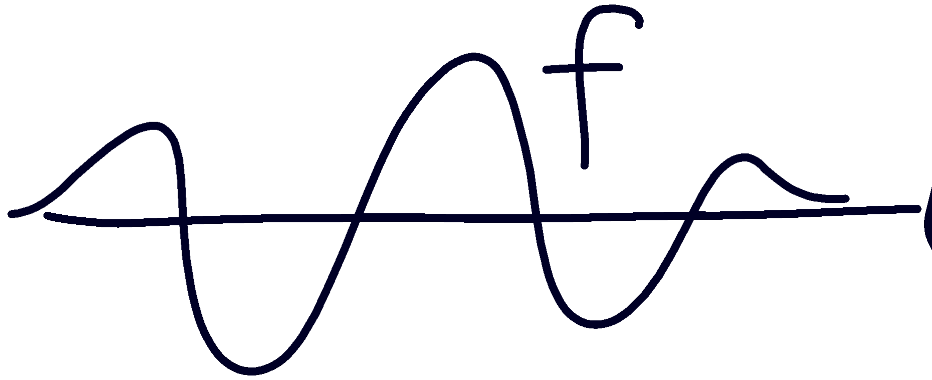
$\cdot$ $\nu \perp \mu$ $\iff$ $|\nu| \perp \mu$ $\iff$ $\nu^+ \perp \mu$, $\nu^- \perp \mu$

Example: $\nu(E) = \int_E f \, d\mu$, denote $d\nu = f\,d\mu$

f: extended μ-integrable

μ: pos. measure

so $f = \frac{d\nu}{d\mu}$



$(X, \mathcal{M}, \mu)$

- $X = P \cup N = \{f > 0\} \cup \{f \leq 0\}$
- $f = f^+ - f^- \Rightarrow \nu(E) = \int_E f^+ d\mu - \int_E f^- d\mu = \nu^+ - \nu^-$
- $|f| = f^+ + f^- \Rightarrow (\nu^+ + \nu^-)(E) = |\nu|(E) = \int_E (f^+ + f^-)\, d\mu = \int_E |f|\, d\mu$

  $\Rightarrow d|\nu| = |f|\, d\mu$
- is $\nu \perp \mu$? Only if $\nu = 0$ (i.e. $f = 0$ μ a.e.)

Rems: 1. $\nu(E) = \int_E (\chi_P - \chi_N)\, d|\nu|$

2. integration w.r.t. signed ν: $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$ $\left(= L^1(|\nu|)\right)$

$\Rightarrow \int f\, d\nu = \int f\, d\nu^+ - \int f\, d\nu^-$

Radon-Nikodym Derivative (3.2)

Q: given $\mu \geq 0$, signed ν on $(X, \mathcal{M})$. When is

$d\nu = f\, d\mu$?

Examples: $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ 1. $F : \mathbb{R} \to \mathbb{R}$, continuously differentiable, increasing

L-S measure for F (from HW #3) $\to$ $\nu = m_F$ $\quad \mu = m$

$$m_F(E) = \int_E F' \, dm \qquad dm_F = F' dm$$

2. $\nu = \delta_{x_0}$, $\mu = m$. Do <u>not</u> have $d\delta_{x_0} = f \, dm$ for any f!

(recall: $\nu \perp m$)

Def: ν signed, $\mu \geq 0$ on $(X, \mathcal{M})$. We say ν is **absolutely continuous** w.r.t. μ, written $\nu \ll \mu$ if $\nu(E) = 0$ for all $E \in \mathcal{M}$ s.t. $\mu(E) = 0$

Exercise:
- $\nu \ll \mu \iff \nu^+ \ll \mu$ and $\nu^- \ll \mu$
- $\nu \ll \mu$ and $\nu \perp \mu \Rightarrow \nu = 0$

Remark: $d\nu = f\,d\mu \Rightarrow \nu \ll \mu$

ext. μ-int.

Thm: if ν is finite $(|\nu|(X) < \infty)$. Then

$\nu \ll \mu \iff \forall \varepsilon > 0, \exists \delta > 0$ s.t.

pos. meas.

$E \in \mathcal{M}, \mu(E) < \delta \Rightarrow |\nu(E)| < \varepsilon$

(recall from HW 3: $f \in L^1 \Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\mu(E) < \delta \Rightarrow \int_E |f| < \varepsilon$)

Pf: $\Longleftarrow$: ✓

Can take $\nu \geq 0$ (consider $|\nu|$, $|\nu(E)| \leq |\nu|(E)$)

$\Longrightarrow$: if not, $\exists\ \varepsilon > 0$ and $E_j \in \mathcal{M}$ s.t.

$\mu(E_j) < 2^{-j}$ and $\nu(E_j) > \varepsilon\ \forall j$

- set $F_k = \bigcup_{j=k}^{\infty} E_j \searrow F = \bigcap_{k=1}^{\infty} F_k$

- $\mu(F_k) \le \sum_{j=k}^{\infty} \mu(E_j) \le \sum_{j=k}^{\infty} 2^{-j} = \frac{1}{2^{k-1}}$

$$\Rightarrow \mu(F) = \lim_{k\to\infty} \mu(F_k) = 0$$

cont. from above

- $\nu(F) = \lim_{k\to\infty} \underbrace{\nu(F_k)}_{\ge \nu(E_k) > \varepsilon} \ge \varepsilon. \Rightarrow \nu \not\ll \mu$

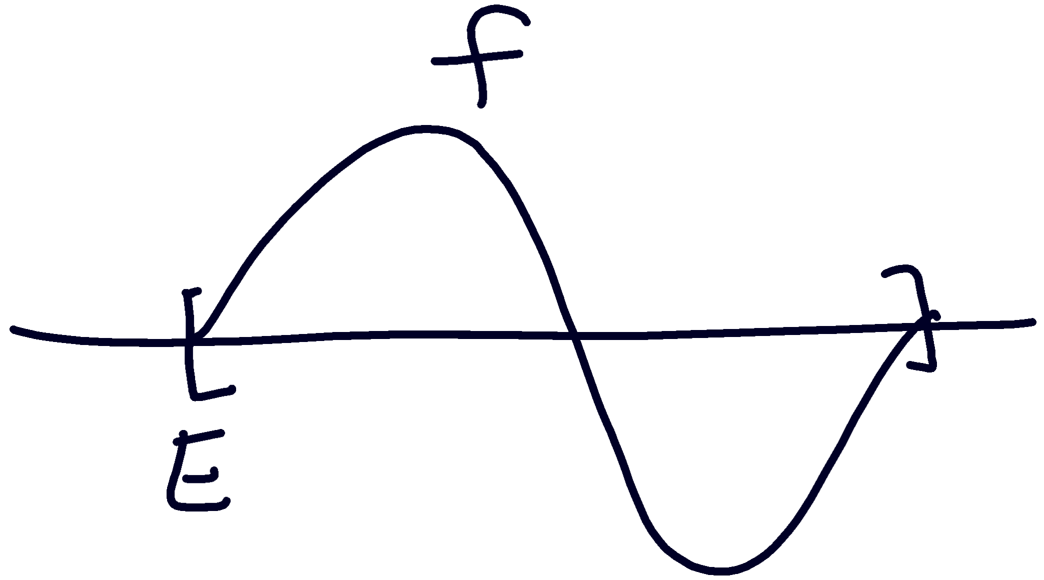
$F_k \supset E_k$

$\blacksquare$

$$|\nu(E)| = |\nu^+(E) - \nu^-(E)|$$

$$\rotatebox{90}{$\geq$}$$

$$|\nu|(E) = \nu^+(E) + \nu^-(E)$$



... coming soon: ν signed, $\mu \geq 0$

$$\cdot\ \nu = \underset{\perp \mu}{\lambda} + \underset{\ll \mu}{\rho} \quad \text{(unique)}$$

$\cdot\ d\rho = f\,d\mu$, $\quad f = \frac{d\rho}{d\mu}$ the "R-N derivative")