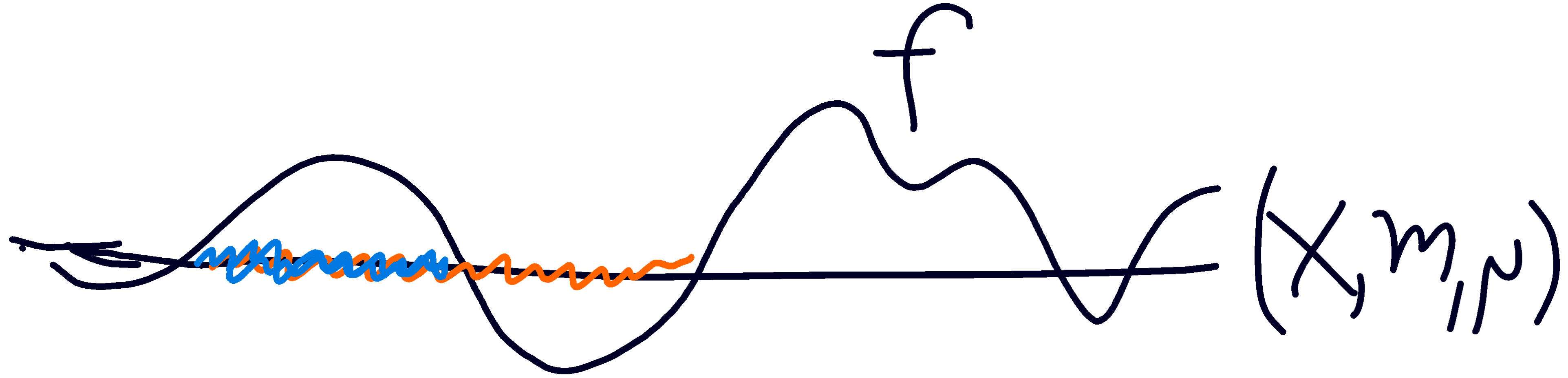
... last time: $(X, \mathcal{M})$

- signed measures: $\nu : \mathcal{M} \to [-\infty, \infty)$ or $(-\infty, \infty]$
- set $E \in \mathcal{M}$ is $\left\{\begin{matrix} \nu\text{-positive} \\ \nu\text{-negative} \\ \nu\text{-null} \end{matrix}\right\}$ if $\left\{\begin{matrix} \nu(F) \geq 0 \\ \nu(F) \leq 0 \\ \nu(F) = 0 \end{matrix}\right\}$ for $\underline{\text{all}}$ $\mathcal{M} \ni F \subset E$

- example: μ a (positive) measure, $f : X \to \mathbb{R}$ extended μ-integrable

$$\nu(E) = \int_E f \, d\mu$$

(written $\underset{\sim\sim\sim}{d\nu = f \, d\mu}$)



Thm (Hahn decomp). ν signed measure on $(X, \mathcal{M})$

$\Rightarrow \exists$ ν-positive P, ν-negative N s.t. $X = P \cup N$, $P \cap N = \emptyset$

(& for any other such P', N'; $P \Delta P'$ ($= N \Delta N'$) is ν-null).

Pf: $\nu : \mathcal{M} \to [-\infty, \infty)$, $\exists$ P, positive, s.t.

$$\nu(P) = m := \sup\{\nu(E),\ E\ \nu\text{-positive}\} < \infty$$

- $N := X \setminus P$
- showed: if $\exists$ $\mathcal{M} \ni E \subset N$ with $\nu(E) > 0$, then $\exists$ $\mathcal{M} \ni F \subset E$ s.t. $\nu(F) > \nu(E)$

- so **if** N is **not** negative, let:
  - n_1 be the 1^{st} integer s.t. $\exists\ A_1 \subset N$ s.t. $\nu(A_1) > \frac{1}{n_1}$
  - n_2 " " 1^{st} integer s.t. $\exists\ A_2 \subset A_1$ s.t. $\nu(A_2) > \nu(A_1) + \frac{1}{n_2}$

$\vdots$

etc.

continuity from above

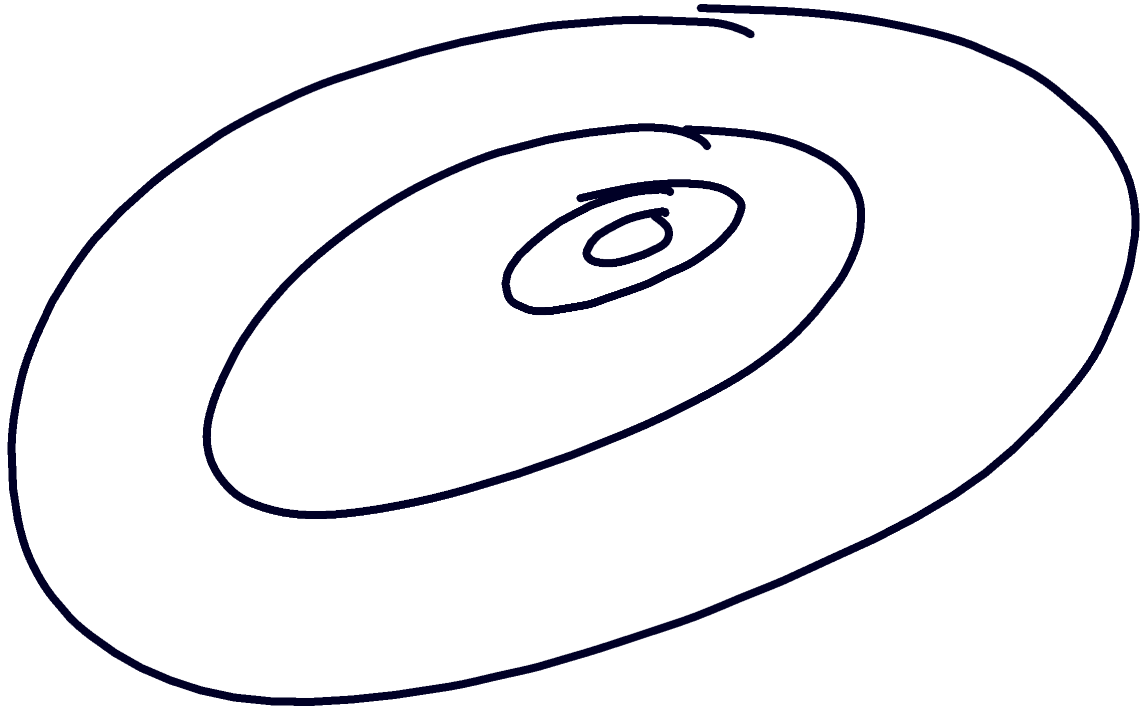
- If $A := \bigcap_{j=1}^{\infty} A_j$, $\infty > \nu(A) = \lim_{j\to\infty} \nu(A_j) \geq \sum_{j=1}^{\infty} \frac{1}{n_j} > 0$

so: $n_j \longrightarrow \infty$.

- but $\exists\ B \subset A$ s.t. $\nu(B) > \nu(A) \Rightarrow \nu(B) > \nu(A) + \frac{1}{n}$

for n large enough, and so if $n_j > n$, $B \subset A_{j-1}$ $\longrightarrow\!\leftarrow$

$\Rightarrow$ N is negative for ν



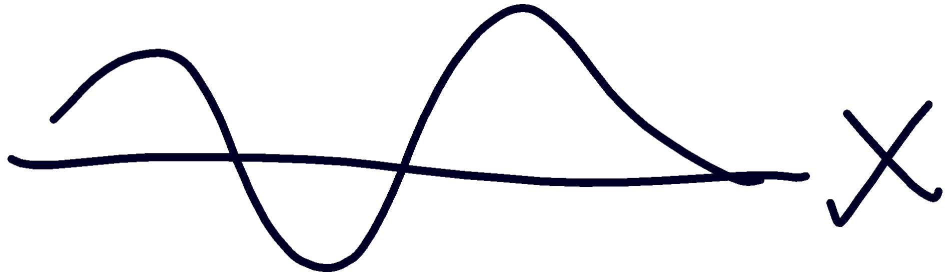
• "uniqueness": if $X = \underset{\text{pos}}{P'} \underset{\text{disjoint}}{\cup} \underset{\text{neg}}{N'}$

$$\Rightarrow \quad P \Delta P' = \underbrace{P \cap N'}_{\substack{\subset P,\, N' \\ \Rightarrow \nu\text{-null}}} \cup \underbrace{N \cap P'}_{\substack{\subset N,\, P' \\ \Rightarrow \nu\text{-null}}} \text{ is } \nu\text{-null}$$

$\blacksquare$

Example: $d\nu = f\,d\mu \Rightarrow P := \{f > 0\}$

$N := \{f \leq 0\}$



Def: signed measure μ, ν on $(X, \mathcal{M})$ are **mutually singular** if $\exists$ $A, B \in \mathcal{M}$ s.t.

$X = A \cup B$, $A \cap B = \emptyset$, B is μ-null, A is ν-null.

Notation: $\mu \perp \nu$

Example: on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$,

- $m \perp \delta_{x_0}$ $\quad \left(\mathbb{R} = \underbrace{\mathbb{R} \setminus \{x_0\}}_{\delta_{x_0}\text{-null}} \cup \underbrace{\{x_0\}}_{m\text{-null}} \right)$

- counting measure $\not\perp$ δ_{x_0}

Thm ("Jordan decomposition"). ν signed measure on $(X, \mathcal{M})$. $\exists$ unique (positive) measures ν^+, ν^- s.t.

$$\nu = \nu^+ - \nu^- \quad \text{and} \quad \nu^+ \perp \nu^-$$

"pos/neg. variations of ν"

Pf: let $X = P \cup N$ be Hahn. $\nu^+(E) := \nu(E \cap P)$
$\nu^-(E) := -\nu(E \cap N)$

- $(\nu^+ - \nu^-)(E) = \nu(E \cap P) + \nu(E \cap P^c) = \nu(E)$ ✓
- P is ν^--null, N is ν^+-null $\Rightarrow$ $\nu^+ \perp \nu^-$ ✓

- if also $\nu = \mu^+ - \mu^-$, $\mu^+ \perp \mu^-$, $X = A \cup B$ disjoint, $\mu^+(B) = \mu^-(A) = 0$

then $X = A \cup B$ is Hahn
A: pos, B: neg

$\Rightarrow$ $A \triangle P$ is ν-null

exercise

- so $\mu^+(E) \overset{\downarrow}{=} \mu^+(E \cap A) = \nu(E \cap A) \overset{\downarrow}{=} \nu(E \cap P) = \nu^+(E \cap P)$
$= \nu^+(E)$

$\Rightarrow \mu^+ = \nu^+$

(so also $\mu^- = \nu^-$).

$\boxed{/\!/}$