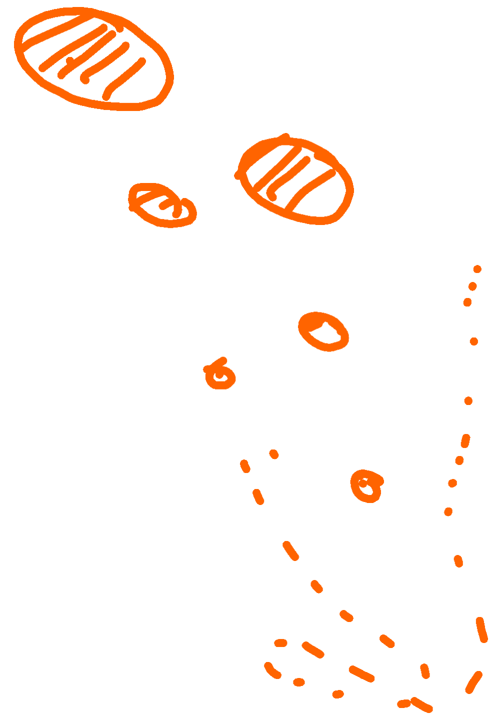
$\boxed{A}$ Measure Theory (esp. Lebesgue Measure on $\mathbb{R}$) (Ch. 1)

$X =$ a (non-empty) set (e.g. $\mathbb{R}$, $[0,1]$, $\mathbb{R}^n$, $\mathbb{Z}$, $\mathbb{Q}$, ...)

$\mathcal{P}(X) = \{E \mid E \subset X\}$ (e.g. $X, \phi \in \mathcal{P}(X)$, $\mathbb{Q} \in \mathcal{P}(\mathbb{R})$, ..)

- a measure, μ of "size" of subsets $E \subset X$ should satisfy:

1. $\mu(\phi) = 0$
2. $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$ if $\{E_j\}_{j=1}^{\infty}$ disjoint
3. $\mu(E) \geq 0$



Examples: 1. $E \subset X$, $\mu(E) = \#\{x \mid x \in E\}$ "counting measure"

2. fix $x_0 \in X$, $\mu(E) = \begin{cases} 1 & x_0 \in E \\ 0 & x_0 \notin E \end{cases}$ "Dirac measure" $\mu = \delta_{x_0}$

What subsets should we measure? All?

Ex: (Folland, p. 20) $\exists$ $E \subset [0,1)$ s.t. with $E_r := E + r \pmod 1$

$\left\{ \begin{array}{l} \cdot\ \{E_r\}_{r \in \mathbb{Q} \cap [0,1)} \text{ disjoint} \\ \cdot\ \bigcup_{r \in \mathbb{Q} \cap [0,1)} E_r = [0,1) \end{array} \right\}$ $\Rightarrow$ cond. 2. is inconsistent with

$\mu([0,1)) = 1$

$\mu(E_r) = \mu(E)$ "tran. inv."

(exercise)

$\Rightarrow$ any generalization of "length" cannot measure <u>all</u> subsets of $\mathbb{R}$

<u>Def:</u> a <u>measure</u> on $(X, \mathcal{M} \subset \mathcal{P}(X)\})$ "measurable sets"

is a fn. $\mu : \mathcal{M} \to [0, \infty]$ satisfying 1.-2. above.

where $\mathcal{M}$ is a: (non-empty)

<u>Def:</u> <u>σ-algebra</u>: closed under
- complement
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(<u>Def:</u> <u>algebra</u>: closed under
- complement
- finite unions)

Exercise:
1. any algebra $\ni \emptyset, X$
2. σ-alg. (alg.) closed under countable $\bigcap$ (finite $\cap$)
3. $\bigcap$ σ-alg.'s is a σ-alg.
4. an algebra closed under countable disjoint $\bigcup$ is a σ-alg.

non-empty

Def: for $\mathcal{E} \subset \mathcal{P}(X)$, the σ-alg. generated by $\mathcal{E}$ is

$$\mathcal{M}(\mathcal{E}) = \text{smallest } \sigma\text{-alg. containing } \mathcal{E}$$
$$= \bigcap \text{ of all } \text{" " "}$$

Example (import): X a topological space. The **Borel σ-alg on X** $\mathcal{B}_X$ is the σ-alg. generated by the open sets.

Rems 1. $\mathcal{B}_{\mathbb{R}}$ contains

- open sets
- closed sets
- countable unions of closed sets ("F_σ sets")
- " $\cap$ " open " ("G_δ sets")

$\vdots$

2. $\mathcal{B}_{\mathbb{R}}$ are generated by any of these familes: $\{(a,b) \mid a<b\}$, $\{[a,b]\}$, $\{(a,b]\}$, $\{[a,b)\}$, $\{(a,\infty)\}$, $\{(-\infty,b)\}$ (exercise).