Measure Theory (esp. Lebesgue Measure on \( \mathbb{R} \)) (Ch. 1)

\( X = \{ \text{a set} \} \) (e.g. \( \mathbb{R}, [a, b], \mathbb{R}^n, \mathbb{Z}, \mathbb{Q}, \ldots \))

\( \mathcal{P}(X) = \{ E | E \subseteq X \} \) (e.g. \( X, \emptyset \in \mathcal{P}(X), \emptyset \in \mathcal{P}(\mathbb{R}), \ldots \))

- A measure \( \mu \) of "size" of subsets \( E \subseteq X \) should satisfy:

1. \( \mu(\emptyset) = 0 \)
2. \( \mu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j) \) if \( \{E_j\} \) disjoint
3. \( \mu(E) \geq 0 \)
Examples:

1. \( ECX, \mu(E) = \# \{ x \mid x \in E \} \) "counting measure".

2. Fix \( x_0 \in X \), \( \mu(E) = \begin{cases} 1 & x_0 \in E \\ 0 & x_0 \notin E \end{cases} \) "Dirac measure" \( \mu = \delta_{x_0} \).

What subsets should we measure? All?

Ex: (Folland, p. 20) \( \exists E \subset [0,1] \) s.t. with \( E_r := E + r \ (\text{mod 1}) \)

\[
\begin{cases}
\forall_r \{ E_r \} \text{ disjoint} \\
\bigcup_{r \in \mathbb{Q} \cap [0,1]} E_r = [0,1]
\end{cases}
\]

\( \Rightarrow \) card. 2. is inconsistent with \( \nu([0,1]) = 1 \)

\( \nu(E_r) = \nu(E) \) \( \text{inv.} \)

(exercise)
Any generalization of “length” cannot measure all subsets of $\mathbb{R}$.

**Def:** a measure on $(X, M \subseteq \mathcal{P}(X))$ "measurable sets" is a fn. $\mu : M \to [0, \infty]$ satisfying 1.-2. above.

where $M$ is a: (non-empty)

**Def:** $\sigma$-algebra: closed under complement

**Def:** algebra: closed under complement, finite unions
Exercise:
1. any algebra $\exists \emptyset, X$
2. $\sigma$-alg. (alg.) closed under countable $\bigwedge$ (finite $\bigvee$)
3. $\bigwedge$ $\sigma$-alg.'s is a $\sigma$-alg.
4. an algebra closed under countable distinct $\bigvee$ is a $\sigma$-alg.

Def: for $E \subset P(X)$, the $\sigma$-alg. generated by $E$ is

$M(E) =$ smallest $\sigma$-alg. containing $E$

$= \bigwedge$ of all $\sigma$-alg. containing $E$
Example (import): $X$ a topological space. The **Borel $\sigma$-alg.** on $X$ is the $\sigma$-alg. generated by the open sets $\mathcal{B}_X$.

**Remarks**

1. $\mathcal{B}_R$ contains:
   - open sets
   - closed sets
   - countable unions of closed sets ("$F_\sigma$ sets")

2. $\mathcal{B}_R$ are generated by any of these families:
   - $\{(a,b) \mid a \leq b\}$
   - $\{[a,b]\}$
   - $\{(a,b]\}$
   - $\{a,\infty\}$
   - $\{(-\infty,b)\}$
   (exercise).