Differentiation of Measures (Ch. 3)

Example: \( f \in L^+ \) and \( \nu : M \to [0, \infty) \)

\[ E \mapsto \int_E f \, d\nu \quad (= \int f X_E \, d\nu) \]

\( X \) notation suggests \( f = \frac{d\nu}{d\nu} \)

Goal: make sense of this!
if $g: X \to [-\infty, \infty]$ is measurable, can define $\nu: M \to [-\infty, \infty]$ with one of

$$E \mapsto \int_E g \, d\nu$$

Then: either

$\text{Ran} \nu \subset [-\infty, \infty)$

or $\text{Ran} \subset (-\infty, \infty]$.

Signed Measures (31)

Def: such a fn. is an extended $\nu$-integrable fn.
**Def.:** a signed measure on \((X, \mathcal{M})\) is a fn. 
\[ \nu: \mathcal{M} \to [-\infty, \infty] \] s.t. 
1. \( \nu(\emptyset) = 0 \)
2. \( \nu \) assumes at most one of \( \infty, -\infty \)
3. if \( \{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M} \) disjoint, then
\[ \nu \left( \bigcup_{j=1}^{\infty} E_j \right) = \sum_{j=1}^{\infty} \nu(E_j) \] (converges absolutely) (if the l.h.s. is finite)
Example: $\nu: E \mapsto \int_E g \, d\mu$ for $g$ extended $\mu$-integrable

$\nu(E) = \int_E (g^+ - g^-) \, d\mu = \int_E g^+ \, d\mu - \int_E g^- \, d\mu$

Goal: $\nu = \nu^+ - \nu^-$

each are (positive) measures
**Def:** $E \in M$ is **positive/negative/null** for $\nu$ if

$\nu(F) \geq 0$, $\nu(F) \leq 0$, $\nu(F) = 0$ for all $M \in \mathcal{F}_E$

**Ex:** $\nu: E \rightarrow \int g \, d\nu_E$  $E$ is positive for $\nu$  

$\Rightarrow g \geq 0 \text{ a.e. on } E$, etc....
Thm ("Hahn Decomposition"). \( \nu \) signed measure on \((X, \mathcal{M})\).

\( \exists P, \text{ positive for } \nu, \; N, \text{ negative for } \nu \text{ s.t. } X = P \cup N \)

\( P \cap N = \emptyset \).

If \( P', N' \) are another such pair,

\( P \Delta P' \; (= N \Delta N') \) is null for \( \nu \).

Proof:

sps \( \nu : M \rightarrow [\mathbb{R}, \mathbb{R}] \) (else take \(- \nu\))

\( m := \sup \{ \nu(E) \mid E \text{ positive for } \nu \} \), so

\( \exists P_j^{\text{pos}} \text{ s.t. } \nu(P_j) \rightarrow m \)
\[ P := \bigcup_{j=1}^{\infty} P_j, \text{ positive for } \nu \]

Exercise/Text: \( P_j \text{ pos } \Rightarrow \bigcup_{j=1}^{\infty} P_j \text{ pos.} \)

\[ m = \lim_{j \to \infty} \nu(P_j) = \nu(P) < \infty \]

Exercise: continuity from above/below holds for signed measures

Set \( N := X \setminus P \). Goal: \( N \) contains no pos. measure subsets

\( N \) cannot contain positive \( E \) with \( \nu(E) > 0 \), since

\[
\nu(P \cup E) = \nu(P) + \nu(E) > m
\]

\( m > 0 \)
- if \( A \cap N, \nu(A) > 0 \), then \\
\( \exists \ B \subset A \) s.t. \( \nu(A) < \nu(B) \), since: \\
\( A \) not positive \( \Rightarrow \exists \ C \subset A, \nu(C) < 0 \), \\
so \( \nu(B := A \setminus C) = \nu(A) - \nu(C) > \nu(A) \)

- next time: contradict \( N \) not negative