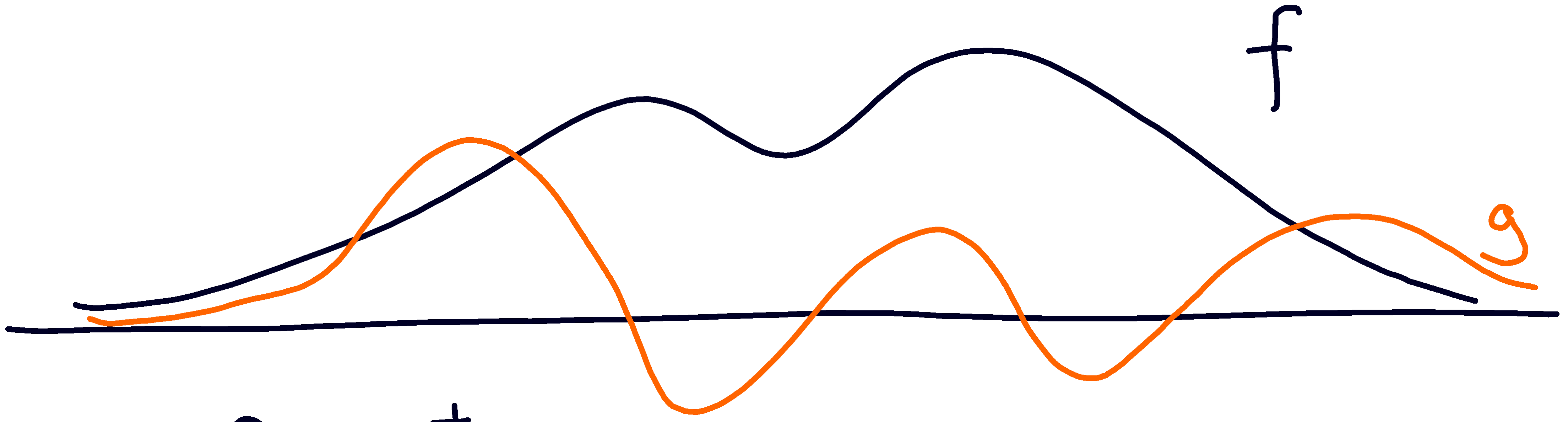
$\boxed{E}$ Differentiation of Measures (Ch. 3)



$(X, \mathcal{M}, \mu)$

Example: $f \in L^+$

$$\nu : \mathcal{M} \longrightarrow [0, \infty]$$

$$E \longmapsto \int_E f \, d\mu \quad \left(= \int_X f \chi_E \, d\mu \right)$$

is another measure on $(X, \mathcal{M})$.

· notation: $d\nu = f \, d\mu$

suggests "$f = \frac{d\nu}{d\mu}$"

Goal: make sense of this!

. if $g : X \to [-\infty, \infty]$ is measurable, can define

$$\nu : \mathcal{M} \to [-\infty, \infty]$$
$$E \mapsto \int_E g \, d\mu$$

$g = g^+ - g^-$ with one of $\int_X g^+ d\mu < \infty$ or $\int_X g^- d\mu < \infty$

Then: either $\operatorname{Ran}\nu \subset [-\infty, \infty)$ or $\operatorname{Ran} \subset (-\infty, \infty]$

Def: such a fn. is an extended μ-integrable fn.

Signed Measures (3.1)

Def: a <u>signed measure</u> on $(X, \mathcal{M})$ is a fn.

$\nu : \mathcal{M} \to [-\infty, \infty]$ s.t.

- $\nu(\emptyset) = 0$
- ν assumes at most one of $\infty, -\infty$
- if $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ disjoint, then

$$\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \nu(E_j)$$

(converges absolutely if the l.h.s. is finite)

Example: $\nu : E \mapsto \int_E g \, d\mu$ for g extended μ-integrable

Note: $\nu(E) = \int_E (g^+ - g^-) \, d\mu = \int_E g^+ d\mu - \int_E g^- d\mu$

each are (positive) measures

Goal: $\nu = \nu^+ - \nu^-$

signed meas.

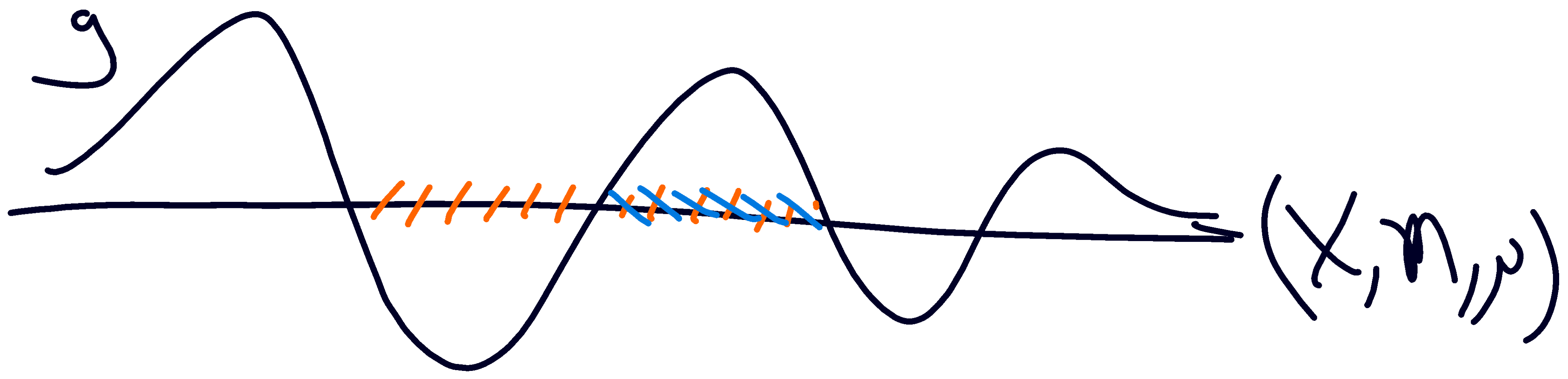
pos. measures

ν signed measure on $(X, \mathcal{M})$

Def: $E \in \mathcal{M}$ is positive / negative / null for ν if

$\nu(F) \geq 0$ / $\nu(F) \leq 0$ / $\nu(F) = 0$ for all $\mathcal{M} \ni F \subset E$

Ex: $\nu : E \mapsto \int_E g \, d\mu$ E is positive for ν $\iff$ $g \geq 0$ μ-a.e. on E, etc......



Thm ("Hahn Decomposition"). ν signed measure on $(X, \mathcal{M})$.

$\exists$ P, positive for ν, N, negative for ν s.t. $X = P \cup N$, $P \cap N = \emptyset$.

If P', N' are another such pair, $P \Delta P'$ $(= N \Delta N')$ is null for ν.

Proof:

- spse $\nu : \mathcal{M} \to [-\infty, \infty)$ (else take $-\nu$)
- $m := \sup\{ \nu(E) \mid E \text{ positive for } \nu \}$, so

$\exists$ P_j^{pos} s.t. $\nu(P_j) \xrightarrow[j \to \infty]{} m$

- $P := \bigcup_{j=1}^{\infty} P_j$, positive for ν

Exercise/text: P_j pos $\Rightarrow \bigcup_{j=1}^{\infty} P_j$ pos.

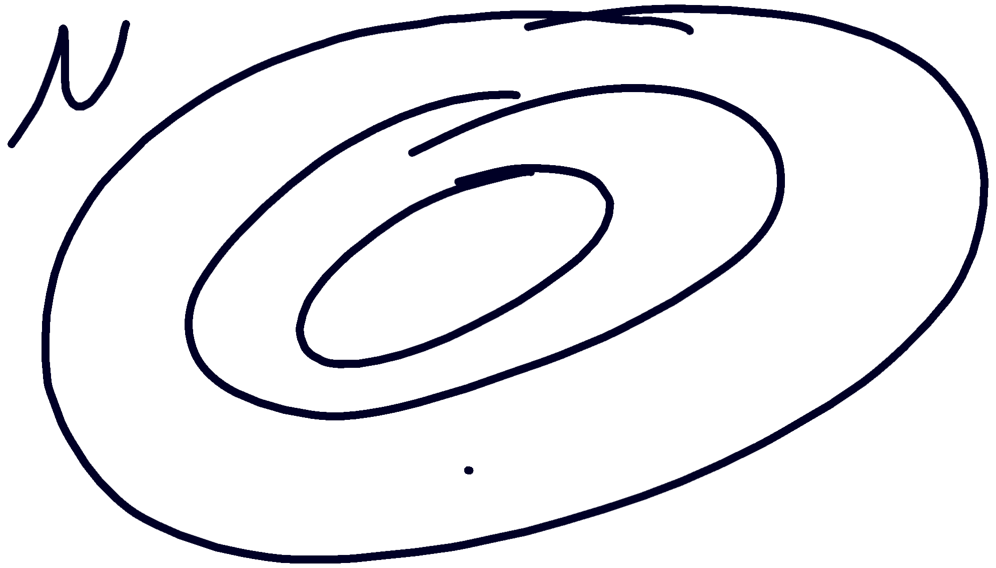
- $m = \lim_{j\to\infty} \nu(P_j) = \nu(P) < \infty$

  Exercise: continuity from above/below holds for signed measures

- set $N := X \setminus P$. Goal: N contains no pos. measure subsets

  - N cannot contain positive E with $\nu(E) > 0$, since then $\nu(\underbrace{P \cup E}_{\text{pos}}) = \underbrace{\nu(P)}_{m} + \underbrace{\nu(E)}_{>0} > m$ $\longrightarrow\!\leftarrow$

N



- if $A \subset N$, $\nu(A) > 0$, then $\exists\ B \subset A$ s.t. $\nu(A) < \nu(B)$, since:

  A not positive $\Rightarrow \exists\ C \subset A$, $\nu(C) < 0$,

  so $\nu(B := A \setminus C) = \nu(A) - \nu(C) > \nu(A)$

- next time: contradict N not negative