$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu) \longrightarrow (X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$

σ-finite

Thm: (Tonelli) $f \in L^+(\mu \times \nu) \Rightarrow$ $x \mapsto \int_Y f_x \, d\nu \in L^+(\mu)$, $y \mapsto \int_X f^y \, d\mu \in L^+(\nu)$,

$$\boxed{\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left[\int_Y f(x,y) \, d\nu(y) \right] d\mu(x) = \int_Y \left[\int_X f(x,y) \, d\mu(x) \right] d\nu(y)} \quad (*)$$

Pf:
- if $f = \chi_E$, $E \in \mathcal{M} \otimes \mathcal{N}$, result is "slicing" theorem above ✓
- if $L^+ \ni f$ is simple, result follows by additivity

- let $0 \le f_n \xrightarrow[n\to\infty]{\text{simple}} f$. Then

$$\underbrace{g_n(x)}_{\in L^+} = \int_Y (f_n)_x \, d\nu(y) \xrightarrow[n\to\infty]{} \int_Y f_x \, d\nu(y) = g(x) \in L^+ \quad \text{by MCT}$$

and $\displaystyle\int_X g_n(x)\, d\mu \xrightarrow[n\to\infty]{} \int_X g(x)\, d\mu$ by MCT. So

$$\int_{X\times Y} f \, d(\mu\times\nu) \underset{MCT}{=} \lim_{n\to\infty} \int_{X\times Y} f_n \, d(\mu\times\nu) = \lim_{n\to\infty} \int_X g_n(x)\, d\mu = \int_X g(x)\, d\mu.$$

Same for $x \leftrightarrow y$. Get $(*)$. $\blacksquare$

<u>Thm:</u> (Fubini) $f \in L^1(\mu \times \nu) \Rightarrow f_x \in L^1(\nu)$ a.e. x
$f^y \in L^1(\mu)$ a.e. y

$g : x \longmapsto \int_Y f_x \, d\nu \in L^1(\mu)$

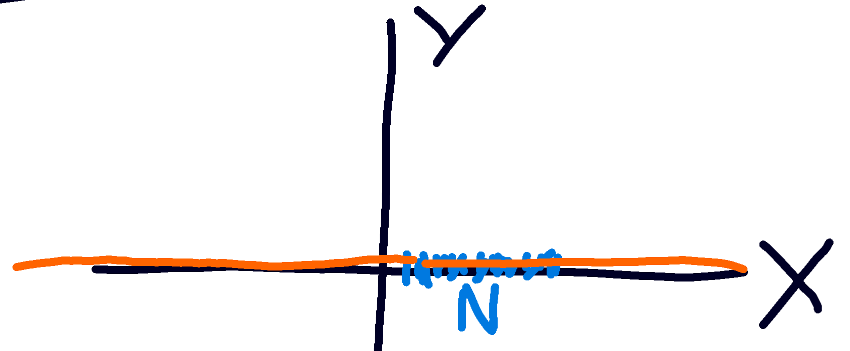
$y \longmapsto \int_X f^y \, d\mu \in L^1(\nu)$ and $\circledast$ holds.

<u>Pf:</u> if $f \geq 0$. Then $\int g \, d\mu \overset{\circledast}{=} \int_{X \times Y} f \, d(\mu \times \nu) < \infty$.
Tonelli $\Rightarrow$

So $g < \infty$ a.e., i.e. $f_x \in L^1(\nu)$ a.e. x, $g \in L^1(\mu)$,
and $\circledast$ holds (same for $x \leftrightarrow y$). For $f : X \times Y \to \mathbb{C} \in L^1(\mu \times \nu)$,
$f = (\mathrm{Re} f)_+ - (\mathrm{Re} f)_- + i\left((\mathrm{Im} f)_+ - (\mathrm{Im} f)_-\right)$. $\boxed{//}$

Rem: how to check $f \in \mathcal{L}^1(\mu \times \nu)$?

$$\int_{X \times Y} \underbrace{|f|}_{\in \mathcal{L}^+} \, d(\mu \times \nu) \underset{\text{Tonelli:}}{=} \int_X \int_Y |f(x,y)| \, d\nu(y) \, d\mu(x) \overset{\text{check}}{<} \infty$$



$N \notin \mathcal{L}$

$N \times \{0\} \notin \mathcal{L} \otimes \mathcal{L}$

$N \times \{0\} \subset \mathbb{R} \times \{0\}, \ (m \times m)(\mathbb{R} \times \{0\}) = 0$

Lebesgue measure on $\mathbb{R}^n$ (2.6)

$$(\mathbb{R}^n, \mathcal{L}^n, m^n) = \text{completion of } (\mathbb{R} \times \cdots \times \mathbb{R}, \ \mathcal{B}_{\mathbb{R}} \otimes \cdots \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^n}, \ m \times \cdots \times m)$$

$$= \text{completion of } (\mathbb{R} \times \cdots \times \mathbb{R}, \ \mathcal{L} \otimes \cdots \otimes \mathcal{L}, \ m \times \cdots \times m)$$

Not'n: sometimes $\mathcal{L}^n = \mathcal{L}$, $m^n = m$

Facts: (all follow easily from facts for $(\mathbb{R}, \mathcal{L}, m)$ — see text):

Thm: (approx. of $\mathcal{L}^n$): $E \in \mathcal{L}^n$

- $m(E) = \inf\{m(U) \mid E \subset U\} = \sup\{m(K) \mid K \subset E\}$
- $E = A_1 \cup N_1 = A_2 \setminus N_2$, A_2 is G_δ, A_1 is F_σ, $m(N_1) = m(N_2) = 0$

- $m(E) < \infty \Rightarrow \forall \varepsilon > 0$, $\exists$ $\{R_j\}_{j=1}^N$ disjoint rectangles whose sides are open intervals with $m\left(E \,\Delta\, \bigcup_{j=1}^N R_j\right) < \varepsilon$

Thm (approximation in L^1) $f \in L^1(m^n)$, $\varepsilon > 0$

- $\exists$ simple $\phi = \sum_{j=1}^{N} a_j \chi_{R_j}$, R_j "classical" rectangles s.t.

$$\int_{\mathbb{R}^n} |f - \phi| \, dm^n < \varepsilon$$

- $\exists$ continuous, compactly supported g s.t. $\int |f-g| < \varepsilon$

Thm: (invariance) m^n is invariant under

- translation: $m^n(E + \underset{\mathbb{R}^n}{a}) = m^n(E)$
- rotation: $T: \mathbb{R}^n \to \mathbb{R}^n$ linear, $T^* T = I$, $m^n(T(E)) = m^n(E)$.