... last time: $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$

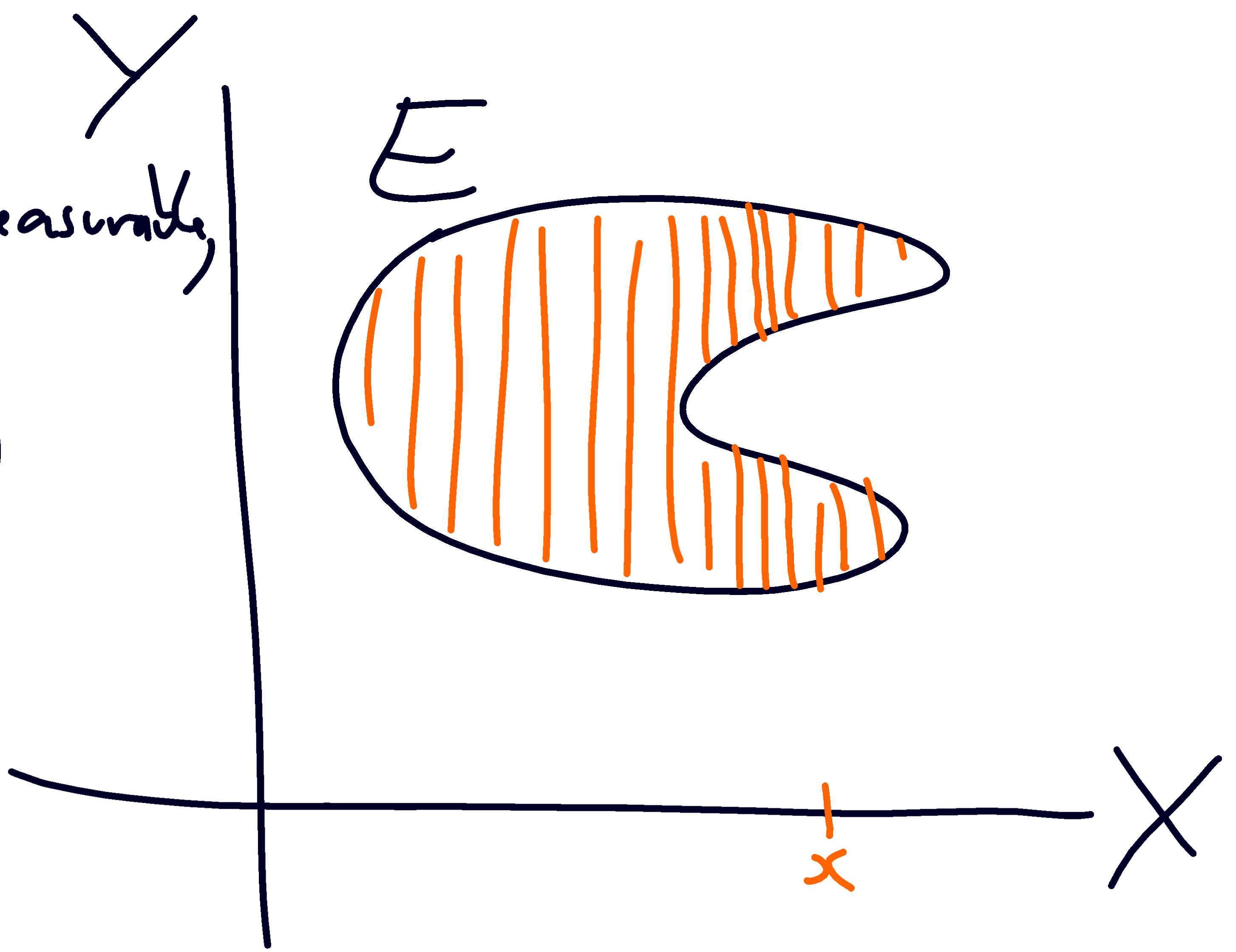
· product measure : $(\mu \times \nu)(A \times E) = \mu(A)\nu(E)$, ... etc.
"rectangle"

$(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$

Thm ("slicing"): $\mu, \nu$ $\sigma$-finite

$E \in \mathcal{M} \otimes \mathcal{N} \Rightarrow x \mapsto \nu(E_x)$, $y \mapsto \mu(E^y)$ measurable,

$$(\mu \times \nu)(E) = \int_X \nu(E_x)\, d\mu(x) = \int_Y \mu(E^y)\, d\nu(y)$$

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$\rightarrow E_x = \{y \in Y \mid (x,y) \in E\} \subset Y$

$E^y = \{x \in X \mid (x,y) \in E\} \subset X$

Exercise: $([0,1], \mathcal{B}_{[0,1]}, m)$ $([0,1], \mathcal{B}_{[0,1]}, \text{counting})$ ← not $\sigma$-finite

$E = \{(x,x) \mid x \in [0,1]\} \subset [0,1]\times[0,1]$

Compute: $\int_{[0,1]} m(E^y)\, d(\text{counting}) \neq \int_{[0,1]} \text{counting}(E_x)\, dm(x) \neq (m \times \text{counting})(E)$

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Pf: · spse $\mu, \nu$ finite

· check for rectangles $E = A \times B$, $A \in \mathcal{M}$, $B \in \mathcal{N}$ (same for $y \leftrightarrow x$)

$\Rightarrow \nu(E_x) = \underbrace{\chi_A(x)\nu(B)}_{\text{meas}} \Rightarrow \int \nu(E_x)\, d\mu(x) = \mu(A)\nu(B) = (\mu\times\nu)(E)$

· holds for $E \in \mathcal{A} = \{$finite disjoint unions of rectangles$\}$ (by additivity)

- $\mathcal{C} := \{ E \in \mathcal{M} \otimes \mathcal{N} \mid$ conclusions of theorem hold $\} \supset \mathcal{A}$ ← algebra

goal: $\mathcal{C} = \mathcal{M} \otimes \mathcal{N}$

- $\mathcal{C}$ is closed under increasing unions:

$\mathcal{C} \ni E_n \nearrow E = \bigcup_{n=1}^{\infty} E_n$. Then $f_n(y) = \mu(E_n^y) \nearrow f(y) = \mu(E^y)$

($f_n$ ← meas.) $\Rightarrow f$ meas.

$$\int_Y \underbrace{\mu(E^y)}_{f} \, d\nu(y) \overset{\text{MCT}}{=} \lim_{n \to \infty} \int_Y \underbrace{\mu(E_n^y)}_{f_n} \, d\nu(y) = \lim_{n \to \infty} (\mu \times \nu)(E_n) \overset{\text{cont. from below}}{=} (\mu \times \nu)(E)$$

(same $x \leftrightarrow y$) $\Longrightarrow E \in \mathcal{C}$

- $\mathcal{C}$ is closed under decreasing intersection:

$$\mathcal{C} \ni E_n \searrow E = \bigcap_{n=1}^{\infty} E_n .$$

$$f(y) \swarrow f_n(y) = \mu(E_n^y) \le f_1(y) = \mu(E_1^y) \in L^1(Y)$$

$f(y)$ (meas)

since $\mu(E_1^y) \le \mu(X)$, $\int_Y \mu(E_1^y) \le \mu(X)\nu(Y) < \infty$

$$\overset{DCT}{\Longrightarrow} \lim_{n\to\infty} \int_Y f_n(y)\,d\nu(y) = \int_Y f(y)\,d\nu(y)$$

$$\lim_{n\to\infty} (\mu\times\nu)(E_n) = (\mu\times\nu)(E)$$

(the second "=" by cont. from above)

$$\Longrightarrow E \in \mathcal{C}$$

**Lemma:** (text)

$\mathcal{A}$ (algebra) $\subset \mathcal{C}$ ("monotone class": closed under countable · decr. $\bigcap$ · incr. $\bigcup$)

$\Rightarrow \mathcal{M}(\mathcal{A}) \subset \mathcal{C}$

- hence: $\mathcal{C} = \mathcal{M} \otimes \mathcal{N}$.
- $\mu, \nu$ σ-finite, consider $X \times Y = \bigcup_{j=1}^{\infty} (A_j \times B_j)$ where $\mu \times \nu (A_j \times B_j) < \infty$

  and apply above to each $A_j \times B_j$

  (details in text)

**Thm** (Tonelli): $f \in L^+(\mu \times \nu)$ | $\mu, \nu$ $\sigma$-finite

$$\Rightarrow \quad x \mapsto \int_Y \underbrace{f_x}_{\in L^+(\nu)} \, d\nu(y) \in L^+(\mu)$$

$$y \mapsto \int_X f^y \, d\mu(x) \in L^+(\nu)$$

$$\int_{X \times Y} f \, d(\mu \times \nu)$$

$$/\!/$$

and $$\int_X \left[ \int_Y f(x,y) \, d\nu(y) \right] d\mu(x) = \int_Y \left[ \int_X f(x,y) \, d\mu(x) \right] d\nu(y)$$