Product Measures, Integration on $\mathbb{R}^n$ (2.5.26)

Let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be measure spaces. Goals:

- Construct product measure

$$\mu \times \nu (E \times A) = \mu (E) \nu (A)$$

- Show:

$$\int f d(\mu \times \nu) = \int \left[ \int f d\nu (y) \right] d\mu (x)$$

- Consider $X = \mathbb{R}$, $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, $\mathcal{M}^n = \mathcal{M} \times \mathcal{M} \times \cdots \times \mathcal{M}$
• start with "rectangles"

\[ M \otimes N \equiv A \times E := \{(x,y) \mid x \in A, y \in B\} \subseteq X \times Y \]

\[ A = \{ \text{finite, disjoint unions of rectangles} \} \]

\[ \sigma - \text{alg. generated by rectangles} \]

text / exercises:

• \( A \) is an algebra

\[ \sigma - \text{alg. gen by } A \text{ is } M \otimes N \]

\[ \mathcal{P}(X \times Y) \]

\[ \mathcal{P} \]

\[ (x,y) \]

\[ x \]

\[ y \]

\[ \bigcup_{j=1}^{n} (A_j \times E_j) \mapsto \bigotimes_{j=1}^{n} \mu(A_j \cap E_j) \]

is a well-defined premeasure

\( (0, \infty = 0) ! \)
Now, from our abstract measure theory (Caratheodory, etc):

\( \mathcal{M} \) generates an outer measure on \( \mathcal{P}(X \times Y) \),

whose restriction to \( M \otimes N \) is a measure extending \( \mathcal{M} \).

**Defn:** this is the product measure \( \mu \times \nu \)

**Rms:**  
- if \( \mu, \nu \) is \( \sigma \)-finite, \( \mu \times \nu \) is \( \sigma \)-finite, so \( \mu \times \nu \) is the unique measure on \( M \otimes N \), extending \( \mathcal{M} \).
- can extend to \( (X_1 \times \cdots \times X_n, M_1 \otimes \cdots \otimes M_n; \mu_1 \times \cdots \times \mu_n) \).
To figure out integration, consider "sections"

\[ E \subseteq X \times Y \]
\[ E_x = \{ y \in Y \mid (x, y) \in E \} \]
\[ E_y = \{ x \in X \mid (x, y) \in E \} \]

For functions: \( f : X \times Y \to \mathbb{C} \)

\[ f_x : Y \to \mathbb{C}, \quad f^y : X \to \mathbb{C} \]
\[ y \mapsto f(x, y), \quad x \mapsto f(x, y) \]
Prop: a) $E \in \mathcal{M} \Rightarrow E_x \in \mathcal{N} \Rightarrow x \in y$

b) $f \text{ } \mathcal{M} \text{-measurable} \Rightarrow f_x \mathcal{N} \text{-meas } x, \ f^y \mathcal{M} \text{-meas } y$

Proof:

a) $R := \{ (x, y) \mid E_x \in \mathcal{N}, E^y \in \mathcal{M} \}$

- $R$ contains all rectangles (exercise)
- $R$ is a $\sigma$-algebra (exercise)

$\Rightarrow R \supset \mathcal{M}$

b) $f^{-1}(B) = \{ (f^{-1}(B)) \} \in \mathcal{N}$ (for $f^y$)
Thm (“slicing”): \((X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)\) \(\sigma\)-finite

\(E \in \mathcal{M} \otimes \mathcal{N}\). Then functions \(x \mapsto \nu(\text{Exp}_x)\) and \(y \mapsto \mu(\text{Eyp}_y)\) are measurable, and

\[\mu \times \nu(E) = \int_X \nu(\text{Exp}_x) \, d\mu(x) = \int_Y \mu(\text{Eyp}_y) \, d\nu(y).\]