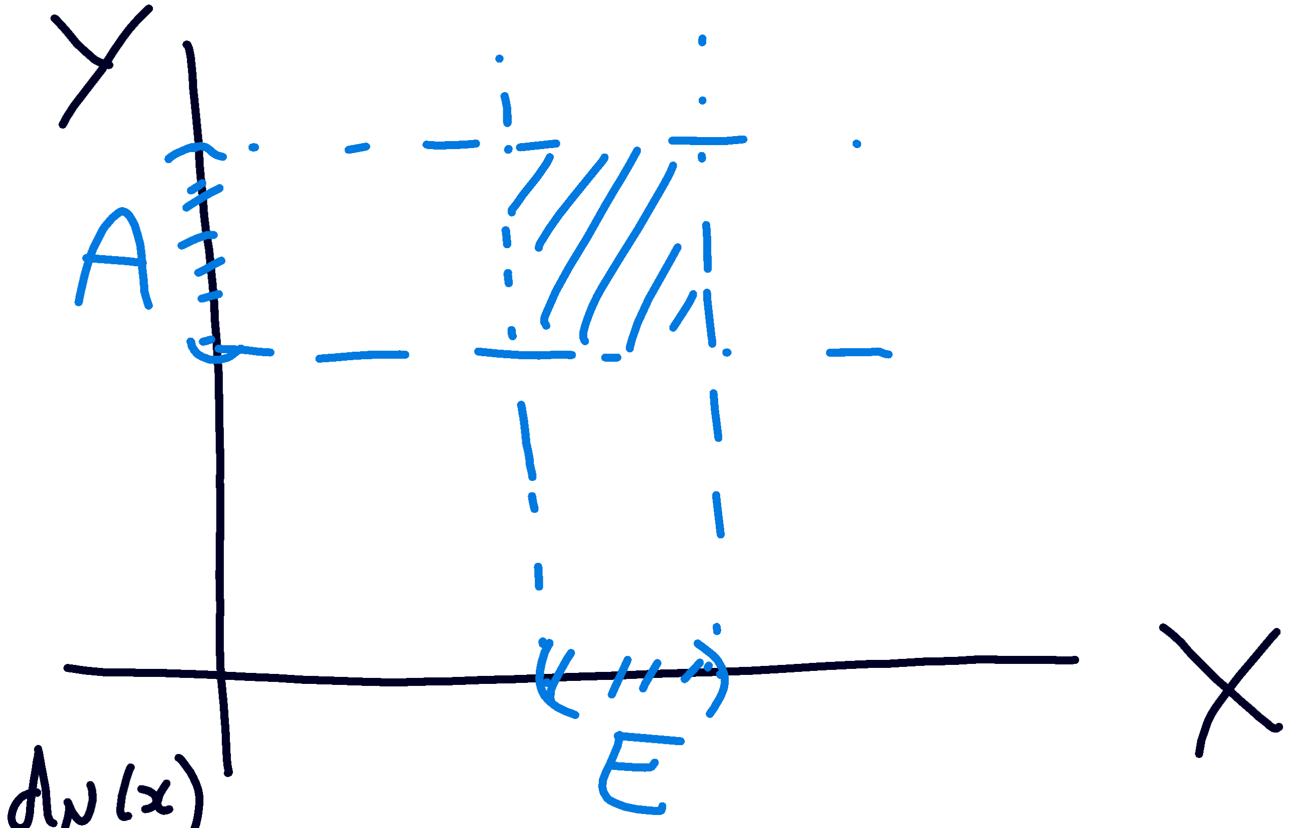
D Product Measures, Integration on $\mathbb{R}^n$ (2.5, 2.6)

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ measure spaces. Goals:



— construct product measure

$$(\mu \times \nu)(E \times A) = \mu(E)\nu(A)$$

— show: $\displaystyle\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left[\int_Y f \, d\nu(y) \right] d\mu(x)$

— consider $X = \mathbb{R}$, $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, $m^n = m \times m \times \cdots \cdots \times m$

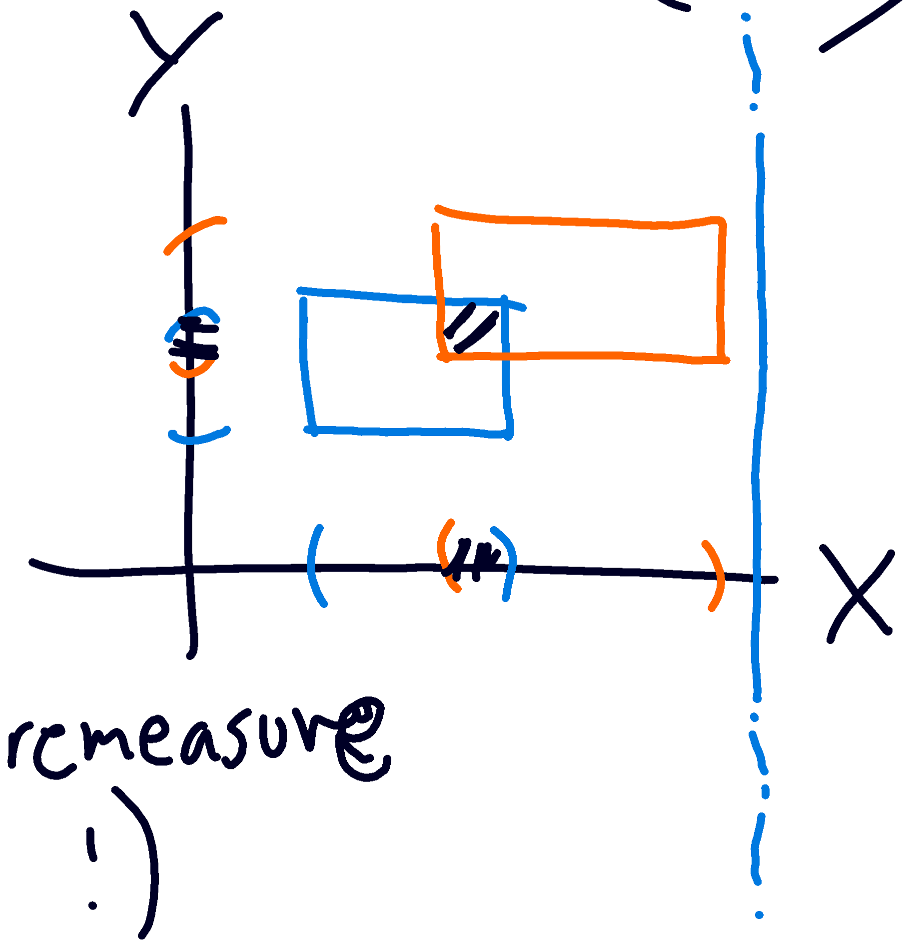
• start with "rectangles"

$\mathcal{M}\otimes\mathcal{N} \ni A\times E := \{(x,y) \mid x\in A, y\in B\} \subset X\times Y$ ⟵

($A \in \mathcal{M}$, $E \in \mathcal{N}$)

($\mathcal{M}\otimes\mathcal{N}$: σ-alg. generated by rectangles)

• $\mathcal{A} = \{$ finite, disjoint unions of rectangles $\} \subset \mathcal{P}(X\times Y)$



text/exercises:
• $\mathcal{A}$ is an algebra
⇝ • σ-alg. gen by $\mathcal{A}$ is $\mathcal{M}\otimes\mathcal{N}$

• $\pi : \mathcal{A} \longrightarrow [0,\infty]$

$$\bigcup_{j=1}^{n}(A_j\times E_j) \longmapsto \sum_{j=1}^{n}\mu(A_j)\nu(E_j)$$

(↑ disj.)

is a well-defined premeasure

$(0\cdot\infty = 0\ !)$

- now, from our abstract measure theory (Carathéodory, etc.):

$\begin{cases} \Pi \text{ generates an outer measure on } \mathcal{P}(X\times Y), \\ \text{whose restriction to } \mathcal{M}\otimes\mathcal{N} \text{ is a measure extending } \Pi \end{cases}$

<u>Defn</u>: this is the <u>product measure</u> $\mu\times\nu$

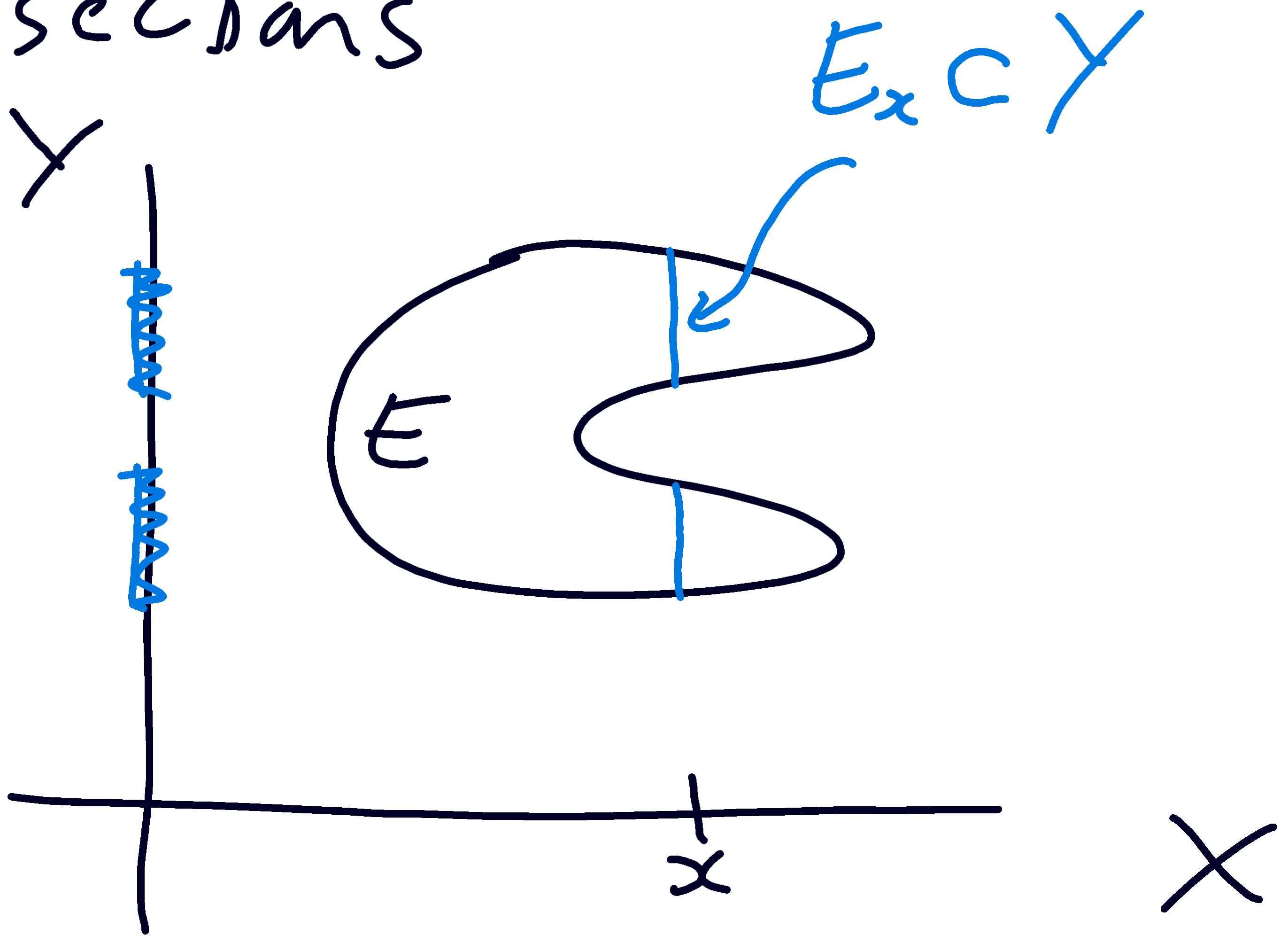
<u>Rems</u>: • if μ, ν is σ-finite, $\mu\times\nu$ is σ-finite, so $\mu\times\nu$ is the <u>unique</u> measure on $\mathcal{M}\otimes\mathcal{N}$, extending Π

- can extend to $(X_1\times\cdots\times X_n, \mathcal{M}_1\otimes\cdots\otimes\mathcal{M}_n, \mu_1\times\cdots\times\mu_n)$

· to figure out integration, consider "sections"

$E \subset X \times Y$ $\quad E_x = \{ y \in Y \mid (x,y) \in E \}$

$E^y = \{ x \in X \mid (x,y) \in E \}$



& for functions: $f : X \times Y \to \mathbb{C}$

$$f_x : Y \to \mathbb{C} \qquad , \qquad f^y : X \to \mathbb{C}$$
$$y \mapsto f(x,y) \qquad\quad x \longmapsto f(x,y)$$

Prop: a) $E \in \mathcal{M} \otimes \mathcal{N} \Rightarrow E_x \in \mathcal{N} \ \forall x \in X$, $E^y \in \mathcal{M} \ \forall y \in Y$

b) f $\mathcal{M} \otimes \mathcal{N}$-measurable $\Rightarrow f_x$ $\mathcal{N}$-meas $\forall x$, f^y $\mathcal{M}$-meas $\forall y$

Pf: a) $R := \{ E \subset X \times Y \mid E_x \in \mathcal{N} \ \forall x, \ E^y \in \mathcal{M} \ \forall y \}$

- R contains all rectangles $\left(\text{eg } (E \times A)_x = \begin{cases} \emptyset & x \notin E \\ A & x \in E \end{cases} \in \mathcal{N} \right)$
- R is a σ-alg (exercise) ✓
- $\Rightarrow R \supset \mathcal{M} \otimes \mathcal{N}$

b) $f_x^{-1}(\underset{\uparrow \text{Borel}}{B}) = \big(\underbrace{f^{-1}(B)}_{\in \mathcal{M} \otimes \mathcal{N}} \big)_x \in \mathcal{N}$ ($\vdots$ for f^y) ✓

Thm ("slicing"): $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ σ-finite

$E \subset \mathcal{M} \otimes \mathcal{N}$. Then functions

$$x \mapsto \nu(E_x) \quad ; \quad y \mapsto \mu(E^y)$$

are measurable, and

$$\mu \times \nu(E) = \int_X \nu(E_x)\, d\mu(x) = \int_Y \mu(E^y)\, d\nu(y).$$