... last time: $(X, \mathcal{M}, \mu)$ $\quad f_n, f : X \to \mathbb{C}$ (measurable)

$f_n \xrightarrow[n\to\infty]{} f$

pointwise: $f_n(x) \to f(x)\ \forall x$

a.e : $f_n(x) \to f(x)\ \forall x \in N^c,\ \mu(N)=0$

uniformly: $\sup_x |f_n(x) - f(x)| \to 0$

L^1 : $\int |f_n - f|\, d\mu \to 0$

measure: $\forall\ \delta > 0,\ \mu(\{x \mid |f_n(x) - f(x)| > \delta\}) \to 0$

$$
\begin{tikzcd}
U \arrow[r] \arrow[rrr, bend left] \arrow[rrrr, dashed, bend left, "\mu(X)<\infty"] & P \arrow[r] & A \arrow[r, dotted, "\mu(X)<\infty"] \arrow[rr, dashed, bend right, "|f_n(x)| \le g(x) \in L^1\ (\text{DCT})"'] & M & L^1 \arrow[l] \arrow[ll, dotted, bend left, "\text{up to subsequence}"]
\end{tikzcd}
$$

Thm (Egoroff): $\mu(X) < \infty$, $\{f_n\}_{n=1}^{\infty}$ (measurable), $f_n \to f$ a.e., then $f_n \to f$ "almost uniformly": for any $\varepsilon > 0$, $\exists$ $E \subset X$, $\mu(E) < \varepsilon$ s.t. $f_n \to f$ uniformly on E^c.

Rem: "almost uniform convergence" $\Longrightarrow$ conv. in measure

(exercise).

So: $\mu(X) < \infty$, $A \longrightarrow M$

Pf: for $k \in \mathbb{N}$, set $E_n(k) := \bigcup_{m=n}^{\infty} \{x \in X \mid |f_m(x) - f(x)| \geq \frac{1}{k}\}$,

decreasing, with $\mu\left(\bigcap_{n=1}^{\infty} E_n(k)\right) = 0$ (by a.e. conv.).

So (cont. from above): $\lim_{n\to\infty} \mu(E_n(k)) = 0$, since $\mu(X) < \infty$.

So for any k, ε, there is n_k s.t. $\mu(E_{n_k}(k)) < \frac{\varepsilon}{2^k}$. Then

$$\mu\left(E := \bigcup_{k=1}^{\infty} E_{n_k}(k)\right) \leq \sum_{k=1}^{\infty} \mu(E_{n_k}(k)) < \varepsilon \sum_{k=1}^{\infty} 2^{-k} = \varepsilon.$$

If $x \notin E$, for $n \geq n_k$ $|f_n(x) - f(x)| < \frac{1}{k}$ $\Rightarrow$ $f_n \to f$ unif. on E^c. $\boxed{/\!/}$

$$\boxed{L^1 \longrightarrow M \xrightarrow[\text{up to subsequence}]{} A}$$

Thm: $\{f_n\}_{n=1}^{\infty}$ "Cauchy in measure": $\forall \varepsilon > 0$, $\lim_{m,n\to\infty} \mu(\{|f_n - f_m| \geq \varepsilon\}) = 0$

then:
- $\exists f$ s.t. $f_n \to f$ in measure
- $\exists$ subsequence $f_{n_j} \longrightarrow f$ a.e.
- if $f_n \to g$ in measure, then $f = g$ a.e.

Rem: if $f_n \to f$ in measure, $\{f_n\}$ is Cauchy in measure

Proof: text

Thm (L^1 approximation of fns). Spse $f \in L^1(\mu)$

- $\forall\, \varepsilon > 0$, $\exists$ simple $\varphi = \sum_{j=1}^{n} a_j \chi_{E_j}$ s.t. $\int |f - \varphi| < \varepsilon$
- $(X, \mu) = (\mathbb{R}, m)$: then
  - can take each E_j to be a finite union of open intervals
  - $\forall\, \varepsilon > 0$, $\exists$ continuous g, with $\underbrace{\text{compact support}}_{\text{vanishing outside an interval}}$

    s.t. $\int |f - g| < \varepsilon$

Pf: · recall $\exists$ simple $\{\varphi_j\}_{j=1}^{\infty}$ s.t.

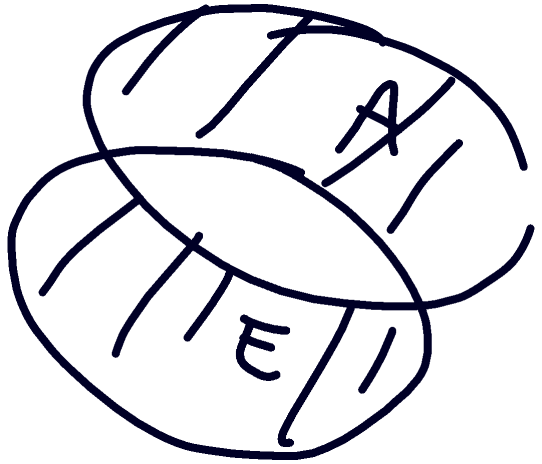
$|\varphi_1| \le |\varphi_2| \le \cdots \le |f|$, $\varphi_j \to f$ pointwise

· then $\lim_{n\to\infty} \int |\varphi_n - f| = 0$ by DCT, since $|\varphi_n - f| \le |\varphi_n| + |f|$

$\le 2|f| \in L^1$

· $(\mathbb{R}, m)$: Prop. 1.20 (Folland): $m(E) < \infty$, $\forall \varepsilon > 0$

$\exists$ A = finite $\cup$ open intervals s.t. $\mu(E \Delta A) < \varepsilon$



? since $\int |\chi_A - \chi_E| = \mu(E \Delta A) < \varepsilon$

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