$\boxed{C}$ Convergence and Approximation of Functions

Modes of convergence (2.4)

- ways in which $f_n : X \longrightarrow \mathbb{C}$, $n = 1, 2, 3, \ldots$ can converge to a limit fn. $f : X \longrightarrow \mathbb{C}$
    - pointwise: $f_n(x) \longrightarrow f(x) \quad \forall\, x \in X$
    - uniform: $\sup_{x \in X} |f_n(x) - f(x)| \longrightarrow 0$

... if now $(X, \mathcal{M}, \mu)$ a measure space:
(f, f_n measurable)

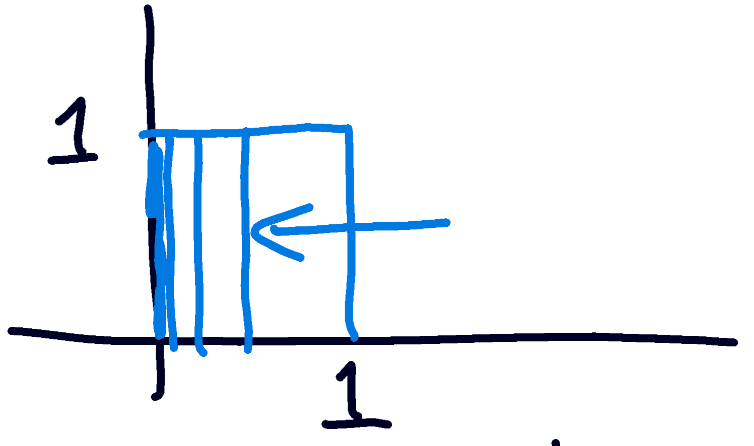
- **almost everywhere**: $f_n(x) \longrightarrow f(x)$ $\forall x \in N^c$, $\mu(N) = 0$
- **L^1**: $\int_X |f_n - f| \, d\mu \longrightarrow 0$
- **in measure**: $\forall \varepsilon > 0$, $\mu\{x \mid |f_n(x) - f(x)| \geq \varepsilon\} \longrightarrow 0$

---

some are stronger than others, eg:

- uniform conv. $\Longrightarrow$ pointwise conv. $\Longrightarrow$ a.e. conv.

$\not\Longleftarrow$ $\chi_{(0,\frac{1}{n})}$ on $\mathbb{R}$ $\qquad$ $\not\Longleftarrow$

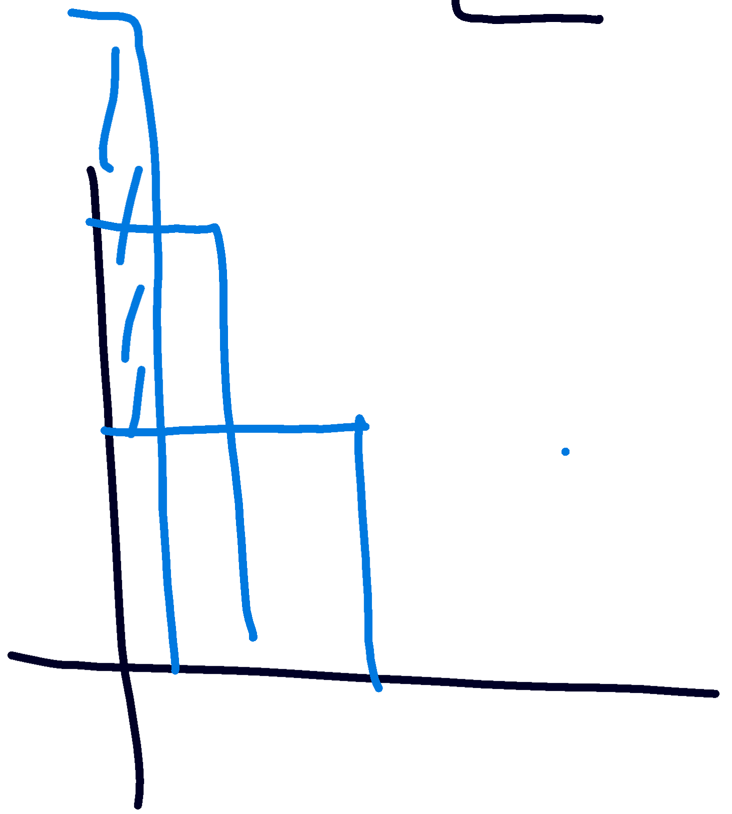


- L^1 conv. $\Longrightarrow$ conv. in meas: $0 \leftarrow \int |f_n - f| \geq \int_{\{|f_n-f|\geq\varepsilon\}} |f_n - f|$

$$\geq \varepsilon\, \mu\{|f_n - f| \geq \varepsilon\}$$

$$\Longrightarrow \mu\{|f_n - f| \geq \varepsilon\} \to 0$$

$\not\Longleftarrow$ $n\chi_{(0,\frac{1}{n})}$ on $\mathbb{R}, m$



so far:

$$\begin{tikzcd} U \arrow[r] \arrow[rrr, bend left] & P \arrow[r] & AE & M & L^1 \arrow[l] \end{tikzcd}$$

(all reverse implications false!)

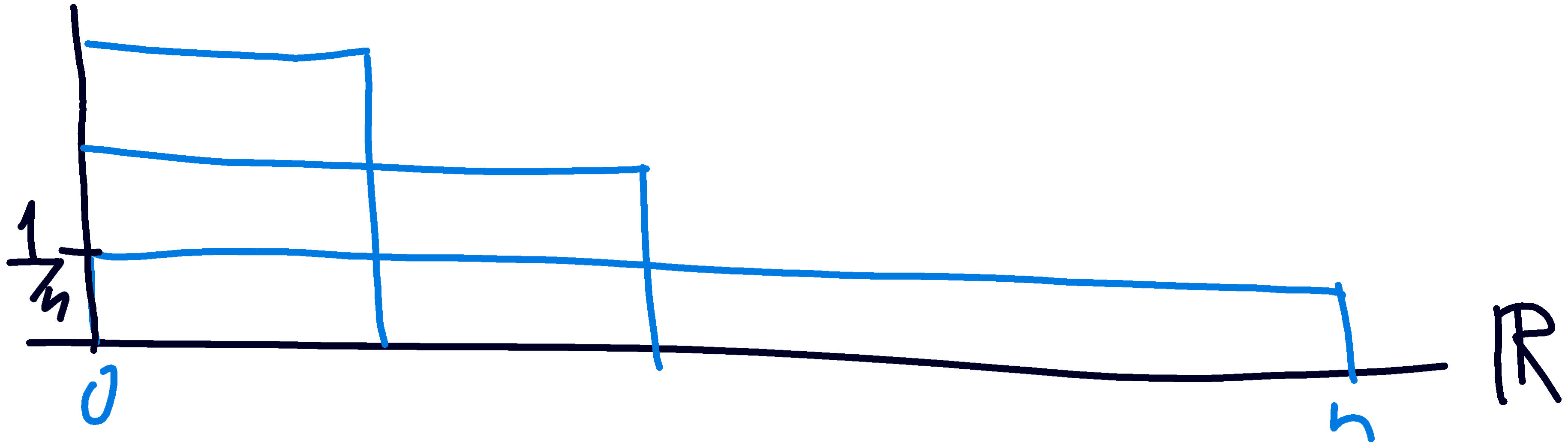
- uniform conv $\Longrightarrow$ conv. in measure: $\forall \varepsilon > 0$ $\{|f_n(x) - f(x)| \geq \varepsilon\} = \emptyset$ for n suff. large

  $\not\Longleftarrow$ $\chi_{(0,\frac{1}{n})}$ on $\mathbb{R}, m$

- some questions:

  1. does unif. conv. $\Longrightarrow$ L^1 conv.?

  No. eg, $\frac{1}{n}\chi_{(0,n)}$



<u>But:</u>

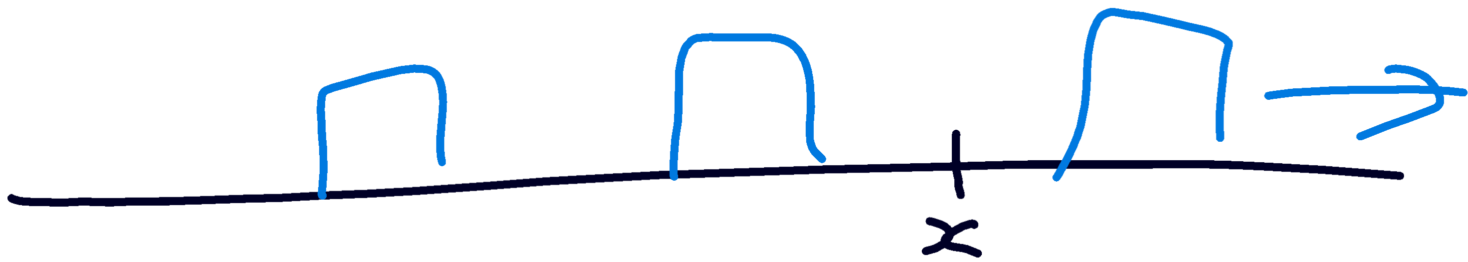
- if $\mu(X) < \infty$, then
$$\int_X |f_n - f| \le \sup_x |f_n(x) - f(x)| \, \mu(X) \longrightarrow 0$$

- if $f_n \to f$ a.e., and $|f_n| \le g \in \mathcal{L}^1$
then DCT $\Longrightarrow \int |f_n - f| \to 0$

Rem: $\frac{1}{n}\chi_{(0,n)} \le \min\left(\frac{1}{x}, 1\right) \longleftarrow$ just fails to be $\in \mathcal{L}^1$

2. does pointwise conv. $\Rightarrow$ conv. in measure?

No. $\chi_{(n,n+1)}$ on $\mathbb{R}$, m

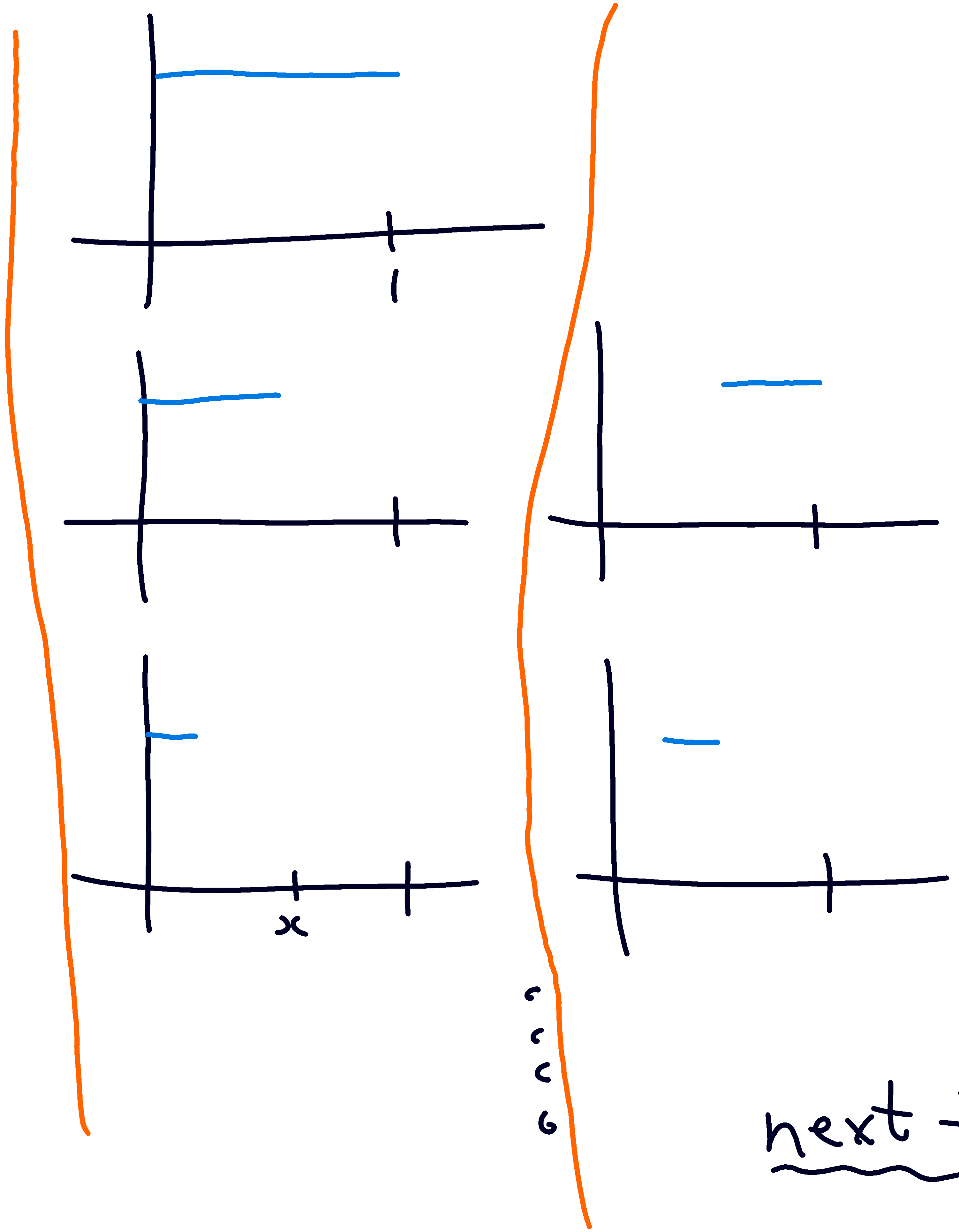


But: if $\mu(X) < \infty$, pwse conv. $\Rightarrow$ conv. in meas. (next time)

3. does L^1 conv. $\Rightarrow$ a.e. conv?

No.

ex:



$$f_n(x) = \chi_{\left[\frac{j}{2^k}, \frac{(j+1)}{2^k}\right]}$$

if $n = 2^k + j$, $j = 0, 1, 2, \dots, 2^k - 1$

$$\int f_n = \frac{1}{2^k} \to 0$$

$f_n \not\to 0$ a.e.

next time: $\exists$ a subsequence $\longrightarrow 0$ a.e.