∘∘∘ so far: $(X, \mathcal{M}, \mu)$

- $L^+ \ni f \mapsto \int f$ linear, monotone
- convergence theorems: $L^+ \ni f_n \to f$ a.e.

MCT : $f_n \nearrow f \Rightarrow \int f = \lim \int f_n$

increasing

Fatou : $\int f \leq \liminf \int f_n$

Integration of complex functions (2.3)

- $f : X \to \mathbb{R}$, measurable

  $f = f^+ - f^-$, $\quad f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$ $\quad f^+ \in L^+$, $f^- \in L^+$

  $\Rightarrow$ set $\int f = \int f^+ - \int f^-$ (if one of $\int f^+, \int f^- < \infty$)

- $f : X \to \mathbb{C}$ measurable: $f = \mathrm{Re} f + i \, \mathrm{Im} f$

  $\int f := \int \mathrm{Re} f + i \int \mathrm{Im} f$ (if well defined)

Def: • $f: X \to \mathbb{C}$ is <u>integrable</u> if $\int \underbrace{|f|}_{\in L^+} < \infty$

• $L^1(\mu) := \{ f : X \to \mathbb{C} \mid \int |f| < \infty \}$

Ex: $\int |f| < \infty \iff \int (\mathrm{Re} f)^{\pm} < \infty, \int (\mathrm{Im} f)^{\pm} < \infty,$

so $\int f$ is defined on L^1

Prop: (a) L^1 is a vector space

(b) $\int$ is a linear map on L^1

$\to$ (c) $f \in L^1 \Rightarrow |\int f| \leq \int |f|$

last line

(d) $f, g \in L^1$, $\int |f-g| = 0 \iff f = g$ a.e. $\iff \int_E f = \int_E g \ \forall E \in \mathcal{M}$

$\int f \chi_E = \int_E f$

Proof: text

Rem: $\int$ does not detect null sets!

→ revised defn:

$$L^1(\mu) = \left\{ \begin{array}{l} \text{equivalence classes of (a.e. defined) integrable fns.} \\ \text{under } f \sim g \Leftrightarrow f = g \text{ a.e.} \end{array} \right\}$$

(makes L^1 a metric (in fact Banach) space under $\int |f-g|$)

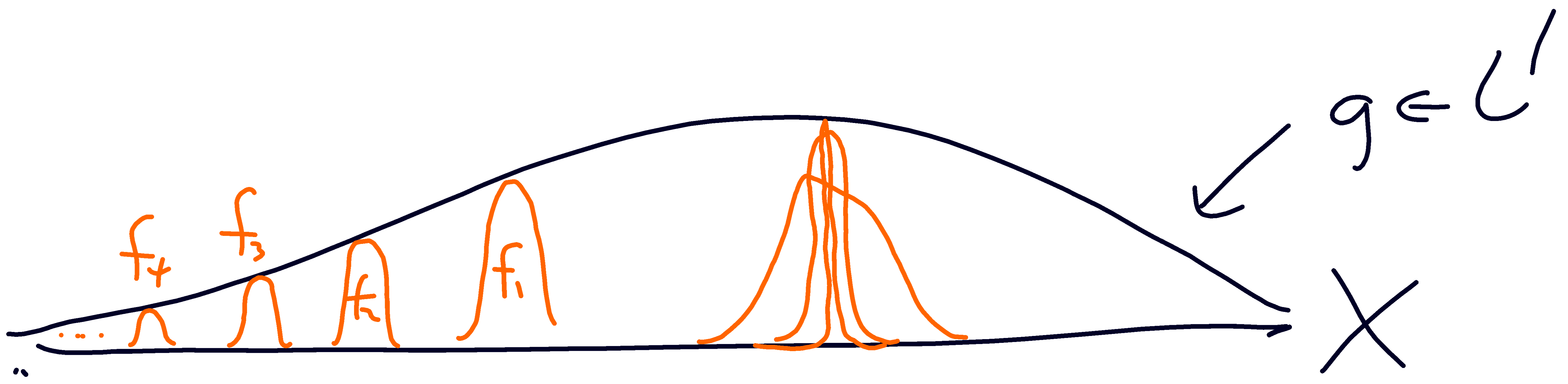
- our final convergence theorem: (does not require $f_n \geq 0$!)

Thm: ("dominated convergence theorem")

- $L^1 \ni f_n \longrightarrow f$ a.e.
- $|f_n(x)| \leq g(x) \in L^1$ for all n

then $\displaystyle\int f = \lim_{n\to\infty} \int f_n$

$f \in L^1$ and



<u>Pf</u>: • $|f_n| \le g \Rightarrow |f| \le g$ a.e. $\Rightarrow f \in \mathcal{L}^1$

• take $f_n \in \mathbb{R}$ (else consider $\operatorname{Re} f_n$, $\operatorname{Im} f_n$)

• $g \pm f_n \ge 0$, so Fatou $\Rightarrow$

$$\int g + \int f = \int g + f \le \liminf \int g + f_n = \int g + \liminf \int f_n$$

$$\int g - \int f = \int g - f \le \liminf \int g - f_n = \int g - \limsup \int f_n$$

$\int g < \infty$, so $\limsup \int f_n \le \int f \le \liminf \int f_n$

$\Rightarrow \lim \int f_n = \int f$ ✓

- application of DCT: exchange $\sum^{\infty}$ and $\int$

<u>Prop:</u> $\{f_j\}_{j=1}^{\infty} \subset L^1$, $\sum_{j=1}^{\infty} \int |f_j| < \infty$

$\Rightarrow \sum_{j=1}^{\infty} f_j$ converges a.e. and

$$\int \underbrace{\sum_{j=1}^{\infty} f_j}_{\in L^1} = \sum_{j=1}^{\infty} \int f_j$$

<u>Pf</u>: • $|f_j| \in L^+$ so $\displaystyle\int \sum_{j=1}^{\infty} |f_j| \underset{\uparrow \text{MCT}}{=} \sum_{j=1}^{\infty} \int |f_j| < \infty$

$$\Rightarrow \sum_{j=1}^{\infty} |f_j| \in L^1$$

<u>Ex</u>: $h \in L^+$, $\int h < \infty \Rightarrow \mu(\{h = \infty\}) = 0$

$$\Rightarrow \sum_{j=1}^{\infty} |f_j(x)| < \infty \text{ a.e.} \Rightarrow \sum_{j=1}^{\infty} f_j(x) \text{ converges a.e.}$$

• since $\displaystyle\left|\sum_{j=1}^{N} f_j(x)\right| \leq \sum_{j=1}^{N} |f_j(x)| \leq \sum_{j=1}^{\infty} |f_j(x)| =: g \in L^1$

• apply DCT to partial sums. ✓

Suggested reading (text)

<u>Thm:</u> $f : [a,b] \longrightarrow \mathbb{R}$, bounded

(a) f Riemann int'ble $\Rightarrow$ f Lebesgue integrable $(\in L^1(m))$

(b) f " " $\Longleftrightarrow m\left(\left\{ \begin{array}{l} x \in [a,b] \text{ where} \\ f \text{ is discontinuous} \end{array} \right\}\right) = 0$

(a.e. continuous)