... previously: $(X, \mathcal{M}, \mu)$

- $f \in L^+$: $\int f := \sup\{ \int \varphi \mid 0 \le \varphi^{\text{simple}} \le f \}$

  — monotone, linear

- MCT : $L^+ \ni f_k \nearrow_{k\to\infty} f \Rightarrow \int f = \lim_{k\to\infty} \int f_k$

//

Prop: $L^+ \ni f$: $\int f = 0 \iff f = 0$ a.e. $\longleftrightarrow \mu(\{x \mid f(x) \neq 0\}) = 0$

"almost everywhere"

Ex: $\int_{\mathbb{R}} \chi_{C_{1/3}}\, dm = 0$, $\int_{\mathbb{R}} \chi_{\mathbb{Q}}\, dm = 0$

Lebesgue

(not Riemann-integrable!)

Pf: • for $f = \sum_{k=1}^{n} a_k \chi_{E_k}$ simple, $a_k \geq 0$

$f = 0$ a.e. $\iff$ for each k, either $a_k = 0$ or $\mu(E_k) = 0$

$\iff \int f = \sum a_k \mu(E_k) = 0$

• $f \in L^+$: $\Leftarrow$ if $f = 0$ a.e.: if $0 \leq \varphi^{\text{simple}} \leq f = 0$ a.e.

$\Rightarrow \varphi = 0$ a.e. $\Rightarrow \int \varphi = 0 \Rightarrow \int f = 0$

$\Rightarrow$: spse not ($f = 0$ a.e.), i.e. $\mu(\{f(x) > 0\}) > 0$.

$E_n := \{f(x) > \frac{1}{n}\}$ $n = 1, 2, \ldots$. $\bigcup_n E_n = \{f(x) > 0\}$

$\Rightarrow 0 < \mu(\{f > 0\}) = \lim_{n\to\infty} \mu(E_n) \Rightarrow \mu(E_k) > 0$, some k

$\Rightarrow f \geq \frac{1}{k} \chi_{E_k} \Rightarrow \int f \geq \frac{1}{k} \mu(E_k) > 0$ ✓

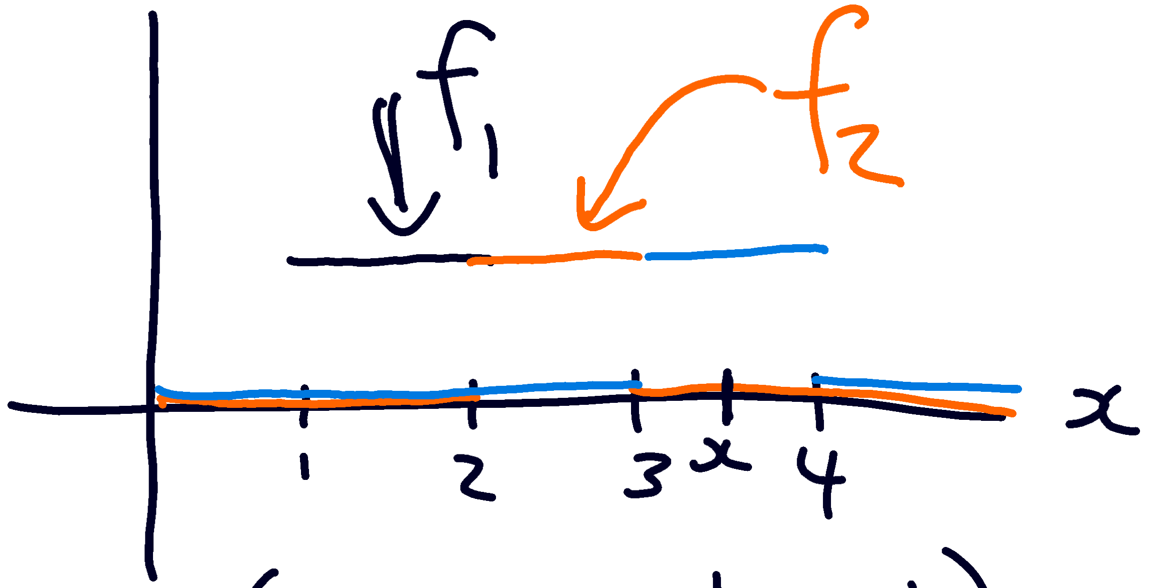
moral: $\int$ doesn't see null sets!

eg: MCT$'$: $\underbrace{\mathcal{L}^+ \ni f_n \nearrow f \text{ a.e.}}_{\mathcal{N}\big(\underbrace{\{x \mid f_n(x) \not\to f(x)\}}_{:= N}\big) = 0} \Rightarrow \int f = \lim_{n\to\infty} \int f_n$

Pf: apply MCT to $f_n \chi_{N^c}$

(.... exercise)

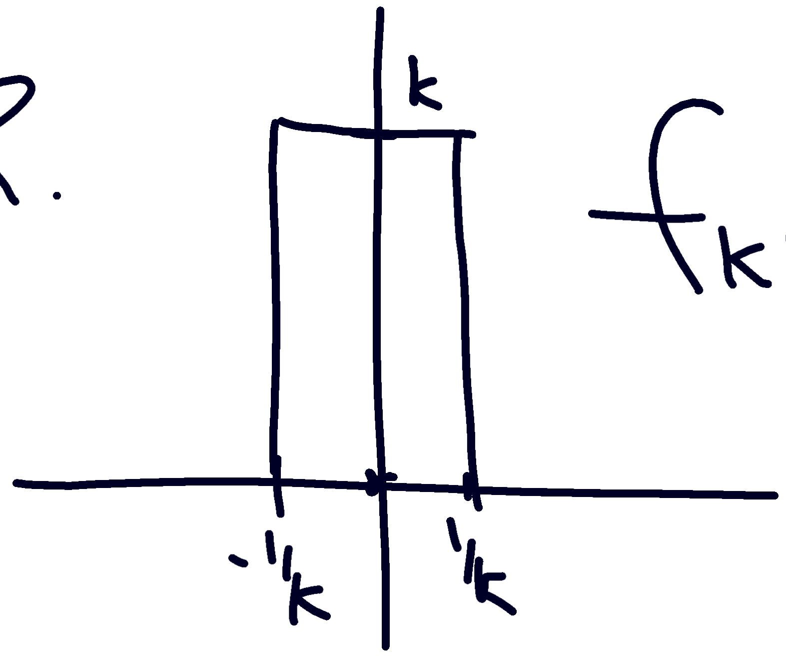
Examples: 1. $f_k(x) = \chi_{[k, k+1]}(x) \in L^+$ $k = 1, 2, 3, \ldots$



(not monotone!)

- $\int_{\mathbb{R}} f_k = m([k, k+1]) = 1 \quad \forall k$ $(\mathbb{R}, \mathcal{L}, m)$

- $\lim_{k \to \infty} f_k(x) = 0$ so $\int 0 \neq \lim \int f_k = 1$

2. 

$$f_k(x) = k \chi_{\left[-\frac{1}{k}, \frac{1}{k}\right]}(x) \xrightarrow{k \to \infty} \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} = 0 \text{ a.e.}$$

- $\int f_k = k \cdot \frac{2}{k} = 2 \ \forall k$ $\quad 0 = \int \lim f_k \neq \lim \int f_k = 2$

3. $f_k(x) = \frac{1}{k} \xrightarrow[k\to\infty]{} 0$

$\int_{\mathbb{R}} f_k = \infty \qquad \int_{\mathbb{R}} 0 = 0$

<u>Rems:</u> in each case:

- f_k not <u>increasing</u>
- $\int \lim f_k < \lim \int f_k$

<u>Thm:</u> "Fatou's Lemma": $\{f_n\}_{n=1}^{\infty} \subset L^+$:

$$\boxed{\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n}$$

<u>Cor:</u> if $f_n \to f$ a.e., $\int f \le \liminf_{n\to\infty} \int f_n$

Pf: $g_k(x) := \inf_{n \geq k} f_n(x)$ are increasing:

MCT $\Rightarrow \int \underbrace{\sup_k g_k}_{\liminf f_n} = \lim_{k\to\infty} \int \inf_{n\geq k} f_n(x)$

- for each $j \geq k$, $\inf_{n \geq k} f_n(x) \leq f_j(x)$

$\Rightarrow \int \inf_{n\geq k} f_n \leq \int f_j$

$\Rightarrow \int \inf_{n\geq k} f_n \leq \inf_{j \geq k} \int f_j$

$$\leq \lim_{k\to\infty} \inf_{j\geq k} \int f_j = \liminf_{n\to\infty} \int f_n \quad \checkmark$$