... last time: integration on a measure space $(X, \mathcal{M}, \mu)$:

"simple"

- $\varphi(x) = \sum_{j=1}^{n} z_j \chi_{E_j}(x) \Rightarrow \int \varphi := \sum_{j=1}^{n} z_j \, \mu(E_j)$ $\quad (E_j \in \mathcal{M})$
  - monotone
  - linear

- $f \in L^+ := \{ f : X \to [0,\infty] \text{ measurable} \} \Rightarrow$

$$\int f := \sup \left\{ \int \varphi \;\middle|\; 0 \le \varphi \le f, \; \varphi \text{ simple} \right\}$$

- monotone
- linear?

Thm (approx. by simple fns)

(a) $f: X \to [0,\infty]$ measurable. $\exists$ simple $0 \le \varphi_1 \le \varphi_2 \le \cdots\cdots \le f$ s.t.

- $\varphi_n \to f$ (pointwise: $\lim\limits_{n\to\infty} \varphi_n(x) = f(x)$)
- $\varphi_n \to f$ uniformly on sets where f bounded

($f(x) \le M$ for $x \in E \Rightarrow \lim\limits_{n\to\infty} \sup\limits_{x\in E} |\varphi_n(x) - f(x)| = 0$)

(b) $f: X \to \mathbb{C}$ measurable: $\exists$ simple $\{\varphi_n\}$ with $0 \le |\varphi_1| \le |\varphi_2| \le \cdots \le |f|$ s.t. $\varphi_n \to f$ pointwise, and unif. on sets where f bounded

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Thm ("monotone convergence"): $\{f_n\} \subset L^+$ with $0 \le f_1 \le f_2 \le f_3 \le \cdots$:

$$\boxed{\int \underbrace{\lim_{n\to\infty} f_n}_{=\sup_n f_n} = \lim_{n\to\infty} \int f_n}$$

Prop: $\{f_n\} \subset L^+$ (finite or infinite): $\int \sum_n f_n = \sum_n \int f_n$

Pf: • $f_1, f_2 \in L^+$: $\exists$ simple $\varphi_n \in L^+$, $\psi_n \in L^+$ $\varphi_n \nearrow f_1$, $\psi_n \nearrow f_2$ (by Thm)

$\Rightarrow \quad \varphi_n + \psi_n \nearrow f_1 + f_2$

$\Rightarrow$ (MCT) $\quad \int f_1 + f_2 = \lim_{n\to\infty} \int (\varphi_n + \psi_n) = \lim_{n\to\infty} \left(\int \varphi_n + \int \psi_n \right)$

$\overset{(MCT)}{=} \int f_1 + \int f_2$

• same for N fns

• $\{f_n\}_{n=1}^{\infty}$: $\quad \sum_{n=1}^{N} f_n \nearrow \sum_{n=1}^{\infty} f_n \quad \underset{MCT}{\Longrightarrow} \quad \int \sum_{n=1}^{\infty} f_n = \int \lim_{N\to\infty} \sum_{n=1}^{N} f_n$

$= \lim_{N\to\infty} \int \sum_{n=1}^{N} f_n = \lim_{N\to\infty} \sum_{n=1}^{N} \int f_n = \sum_{n=1}^{\infty} \int f_n$ ✓

Pf. of approx thm:

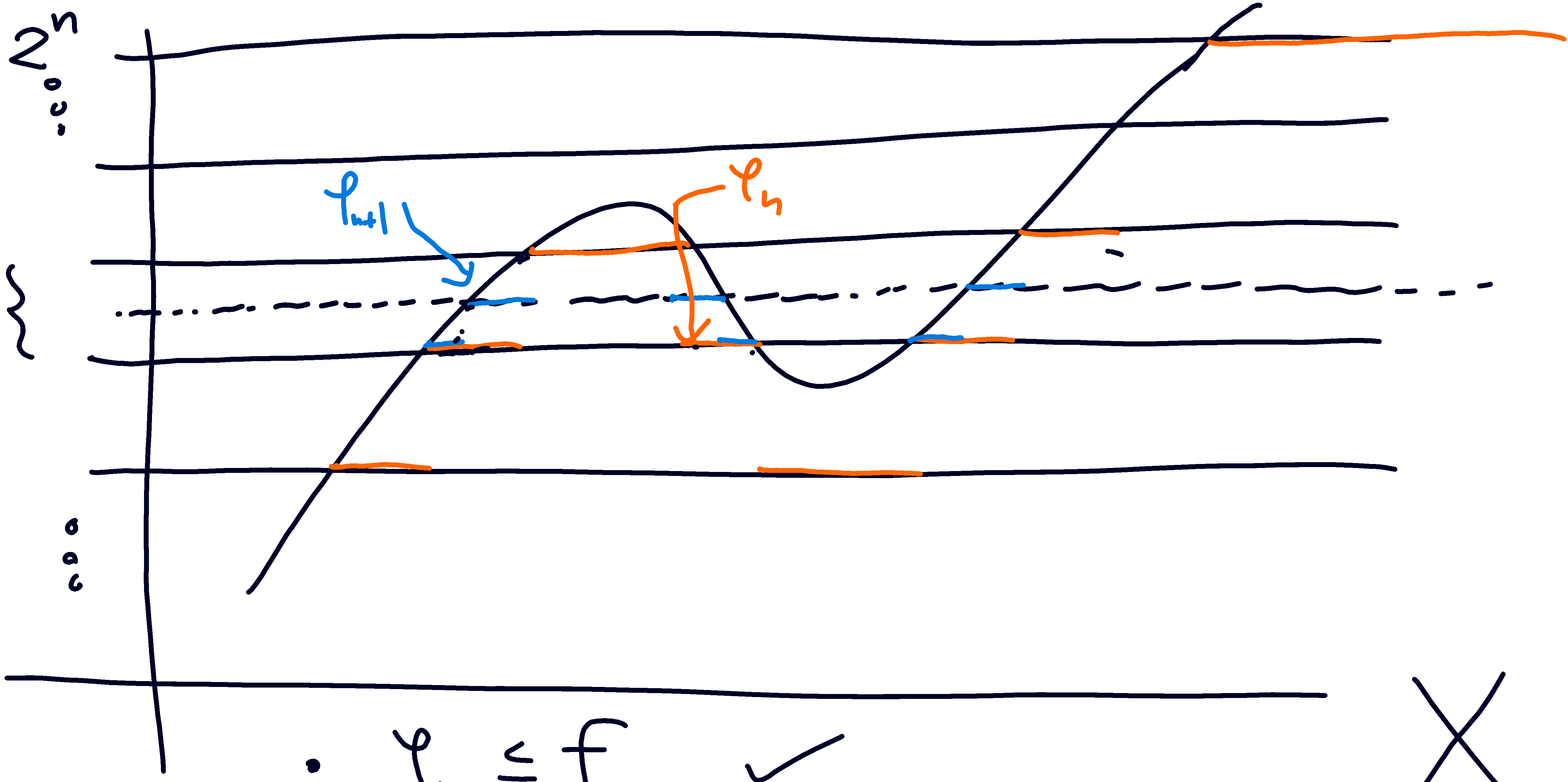
$n = 1, 2, 3, \ldots\ldots$

$E_n^k := f^{-1}\left(\left(k2^{-n}, (k+1)2^{-n}\right]\right)$

$0 \le k \le 2^{2n} - 1$

$F_n = f^{-1}\left((2^n, \infty]\right)$

$$\varphi_n(x) = \sum_{k=0}^{2^{2n}-1} k2^{-n} \chi_{E_n^k}(x) + 2^n \chi_{F_n}(x)$$



- $\varphi_n \le f$ ✓
- $\varphi_n \le \varphi_{n+1}$ ✓
- $f(x) \le 2^n \implies 0 \le f(x) - \varphi_n(x) \le 2^{-n} \xrightarrow[n \to \infty]{} 0$

  (if $f(x) = \infty$, $\varphi_n(x) = 2^n \longrightarrow \infty$)

(b) $f = g + ih$

$g = g^+ - g^-$

$h = h^+ - h^-$

and proceed.

exercise ✓

Proof of MCT: · $f(x) := \sup_n f_n(x) \in L^+$

· $\{\int f_n\}$ increasing $\Rightarrow \lim_{n\to\infty}\int f_n = \sup_n \int f_n$ $\exists$

· $f_n \le f \underset{\text{(monotonicity)}}{\Longrightarrow} \int f_n \le \int f \Rightarrow \lim_{n\to\infty}\int f_n \le \int f$

· if $0 \le \varphi^{\text{simple}} \le f$, fix $\alpha \in (0,1)$. Set

$$\mathcal{M} \ni E_n = \{x \mid f_n(x) \ge \alpha\varphi(x)\}$$

$$E_1 \subset E_2 \subset \cdots\cdots. \quad \bigcup_{n=1}^{\infty} E_n = X$$

· since $E \mapsto \int_E \varphi$ is measure, $\int_{E_n}\varphi \longrightarrow \int \varphi$

$$\int f_n \ge \int_{E_n} f_n \ge \alpha\int_{E_n}\varphi$$

$$\Rightarrow \lim \int f_n \ge \alpha \int \varphi$$

· take $\alpha \nearrow 1$

$$\Rightarrow \lim \int f_n \ge \int \varphi \quad ✓$$