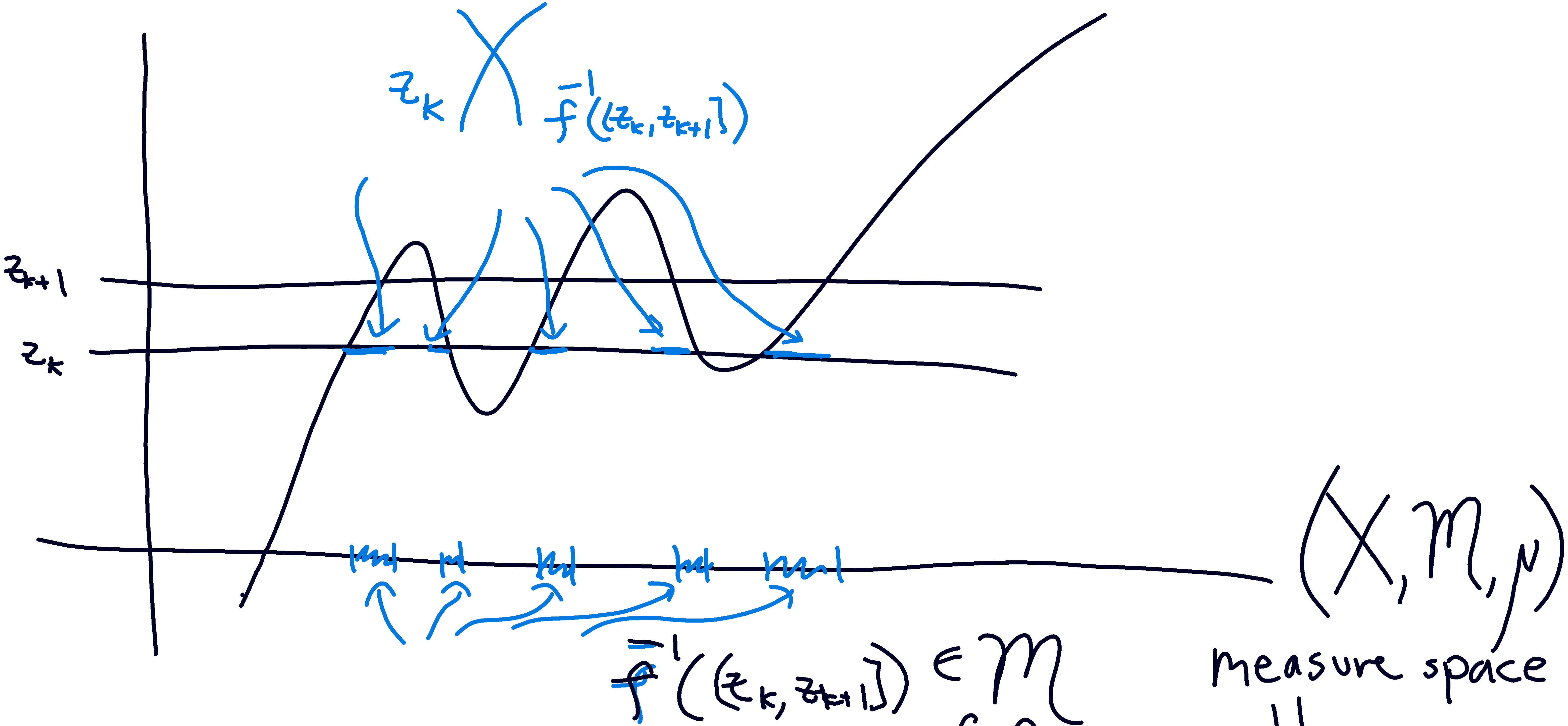
Integration: simple, and non-negative functions (2.2)



$f^{-1}([z_k, z_{k+1}]) \in \mathcal{M}$
if f is measurable

Def: a simple fn. on $(X, \mathcal{M})$ is of the form

$$f(x) = \sum_{j=1}^{n} z_j \chi_{E_j}(x), \qquad z_j \in \mathbb{C}, \quad E_j \in \mathcal{M}$$

Rems:

- f is simple $\iff$ $\operatorname{Ran} f = \{z_1, \dots, z_n\}$ is finite
- f is in "standard form" if $E_j = f^{-1}(\{z_j\})$

  (z_j distinct, $\bigcup_{j=1}^{n} E_j = X$, disjoint)

  ($\chi_{[0,2]} = \chi_{[0,1]} + \chi_{(1,2]}$)

  standard

- sums, products of simple are simple

Def: $(X, \mathcal{M}, \mu)$ measure space, $f : X \to \mathbb{C}$ simple

$$f = \sum_{j=1}^{n} z_j \chi_{E_j} \qquad \boxed{\int f = \sum_{j=1}^{n} z_j \mu(E_j)}$$

Rems:
- convention "$0 \cdot \infty = 0$"
- exercise: check the value is the same for different representations of f

Prop: $\forall$ φ, ψ simple

a) $c \in \mathbb{C} \Rightarrow \int c\varphi = c \int \varphi$

b) $\int \varphi + \psi = \int \varphi + \int \psi$

$\Big\}$ linearity

c) $\varphi, \psi \in \mathbb{R}$: $\underline{\varphi \leq \psi} \Rightarrow \int \varphi \leq \int \psi$

d) $\varphi \geq 0$: $\mathcal{M} \ni A \longmapsto \int_A \varphi := \int \chi_A \varphi$ is a measure

Pf: a) ✓ b) text/exercise

c) $\varphi = \sum_{j=1}^{n} \underset{\mathbb{R}}{z_j} \chi_{E_j} \leq \psi = \sum_{k=1}^{m} \underset{\mathbb{R}}{w_k} \chi_{F_k}$, standard form

$$\int \varphi = \sum_j z_j \mu(E_j) = \sum_j z_j \sum_k \mu(E_j \cap F_k) = \sum_j \sum_k z_j \mu(E_j \cap F_k)$$

$\leq w_k$ if $E_j \cap F_k \neq \phi$

$$\leq \sum_k w_k \sum_j \mu(E_j \cap F_k) = \sum_k w_k \mu(F_k) = \int \psi$$

✓

d) exercise/text

Def: $L^+ := \{ f : X \to [0,\infty] \text{, measurable} \}$

$f \in L^+$: $\int f := \sup \left\{ \int \varphi \;\middle|\; \begin{array}{l} 0 \le \varphi \le f \\ \varphi \text{ simple} \end{array} \right\}$

Rems:

- if f simple, defn. agrees with above one
- $L^+ \ni f \le g \Rightarrow \int f \le \int g$
- $f \in L^+$, $c \ge 0 \Rightarrow \int cf = c \int f$
- additivity is more subtle

need:
- f well-approximated by simple fns
- $\int$ behaves well w.r.t. convergence of fns

next time: 2 Theorems: ("monotone convergence theorem")