Integration: simple, and non-negative functions (2.2)

If $f$ is measurable in $(X, M, \mu)$, then $f^{-1}(z_k, z_{k+1}) \in M$.
Def: a simple fn. on $(X, M)$ is of the form

$$f(x) = \sum_{j=1}^{n} z_{j} X_{E_{j}}, \quad z_{j} \in \mathbb{C}, \quad E_{j} \in M$$

Rems:  
- $f$ is simple $\iff$ $\text{Ran} f = \{z_{1}, \ldots, z_{n}\}$ is finite
- $f$ is in "standard form" if $E_{j} = f^{-1}(\{z_{j}\})$
  - $z_{j}$ distinct, $\cap_{j=1}^{n} E_{j} = X$, disjoint
  - $X_{[0,1]} = X_{[0,1]} \cdot X_{[1,2]}$ (standard)
- sums, products of simple are simple
Def: \((X, M, \mu)\) measure space, \(f: X \to \mathbb{C}\) simple

\[
f = \sum_{j=1}^{n} \mathbb{1}_{E_j} \in L^1(M, \mu)
\]

\[
\int f = \sum_{j=1}^{n} \mathbb{1}_{E_j} \mu(E_j)
\]

Rems: 
- convention "\(0 \cdot \infty = 0\)"
- exercise: check the value is the same for different representations of \(f\)

\(\varphi, \psi\) simple

Prop: \(\land\)

a) \(c \in \mathbb{C} \Rightarrow \int c \varphi = c \int \varphi\) \(\text{linearity}\)

b) \(\int \varphi + \psi = \int \varphi + \int \psi\)

c) \(\varphi, \psi \in \mathbb{R} : \varphi \leq \psi \Rightarrow \int \varphi \leq \int \psi\)

d) \(\varphi \geq 0 : M \ni A \mapsto \int \varphi : \text{is a measure}\)
Pf:  

(a) / text/exercise

(b) text/exercise

(c) \[ Y = \sum_{j=1}^{n} z_j X_{E_j} \leq \psi = \sum_{k=1}^{m} w_k X_{F_k}, \]

standard form

\[ \sum_{j}^{n} \mu(E_j) = \sum_{j}^{n} \sum_{k} \mu(E_j \cup F_k) = \sum_{j}^{n} \sum_{k} \mu(E_j) \rightarrow \leq w_k \text{ if } E_j \cup F_k \neq \emptyset \]

(d) exercise/text
Def: \( L^+ = \{ f : X \to [0, \infty], \text{measurable} \} \)

\( f \in L^+ : \quad \int f = \sup \{ \int g \mid 0 \leq g \leq f \}\)

Rems:

. if \( f \) simple, defn. agrees with above one
. \( L^+ \cap f \leq g \Rightarrow \int f \leq \int g \)
. \( f \in L^+ , c \geq 0 \Rightarrow \int cf = c \int f \)
. additivity is more subtle
need: \( f \) well-approximated by simple \( f \)ns

\( \hat{f} \) behaves well w.r.t. convergence of \( f \)ns

next time: 2 Theorems: ("monotone convergence theorem")