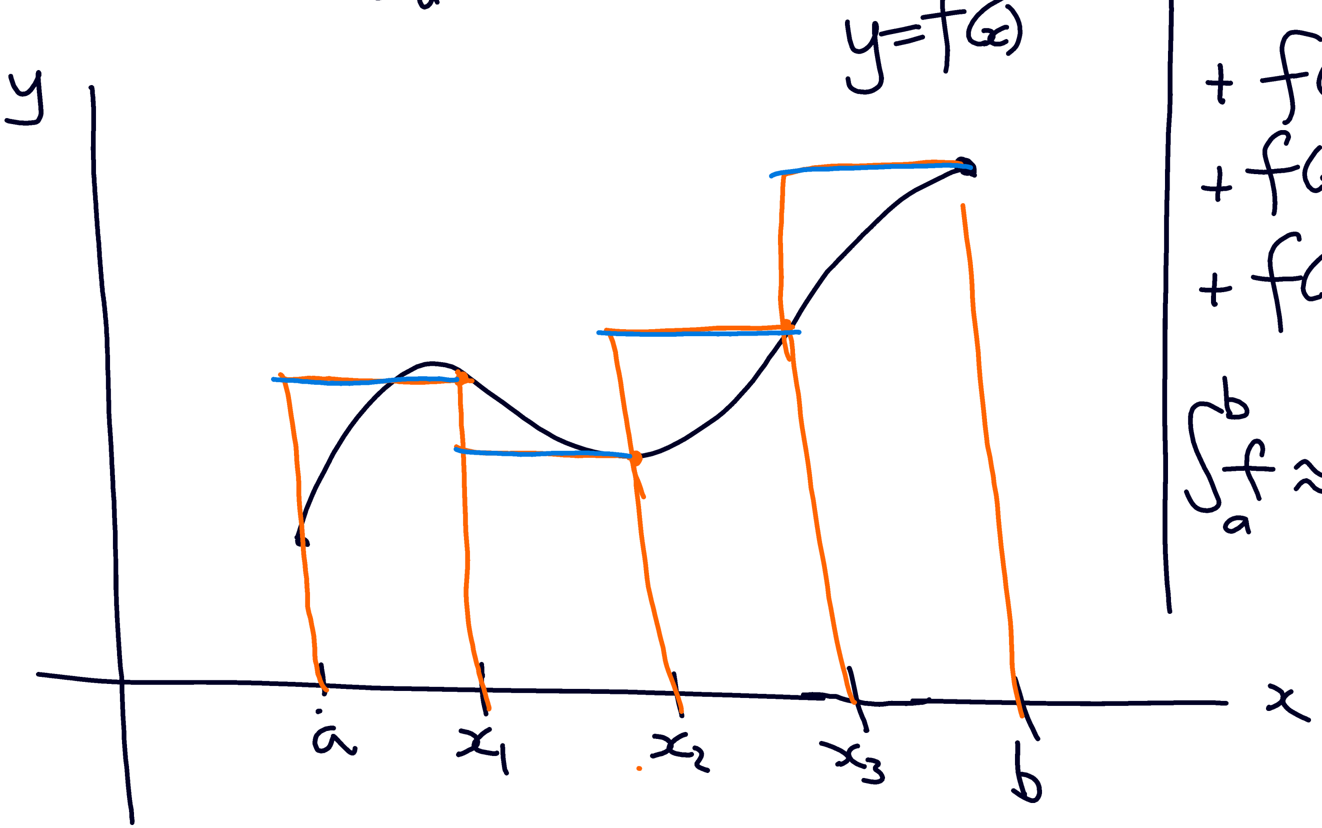
# MATH 420/507

- $\underbrace{\text{measure theory}}_{\text{"size of sets" (length, area, volume, ...)}}$, $\underbrace{\text{integration}}$, ( convergence of functions, differentiation )

$$\chi_E(x) := \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

- which fns. are Riemann integrable?
  - piecewise continuous

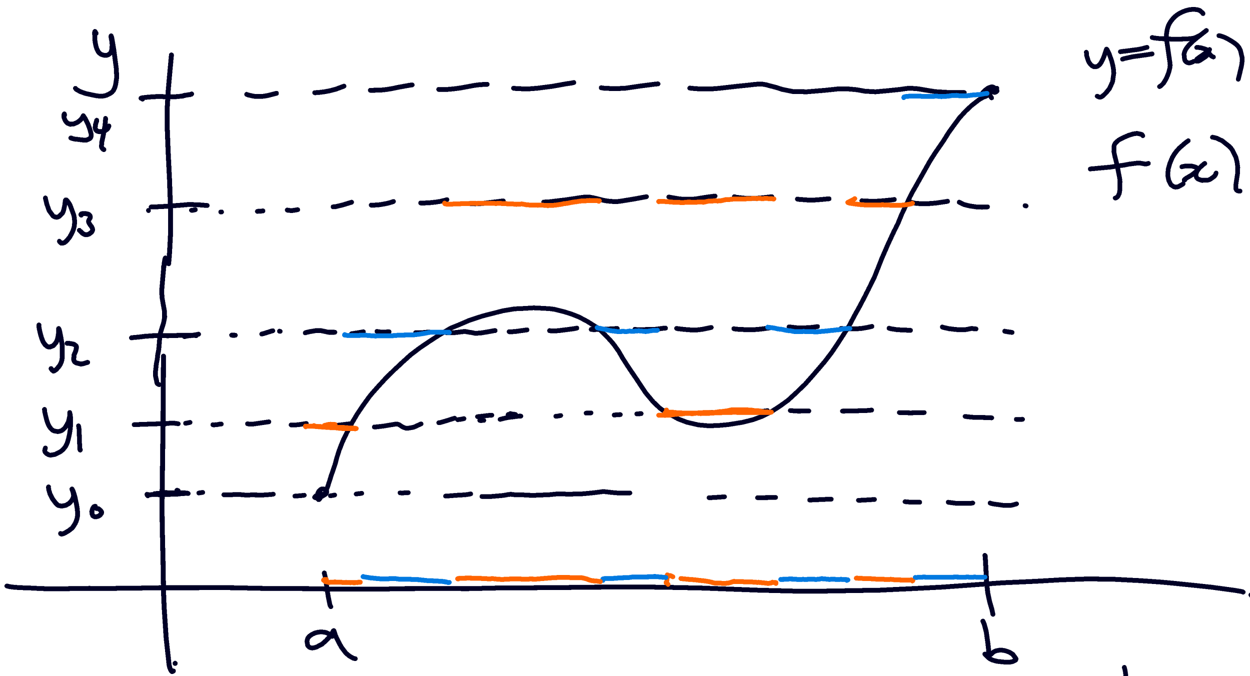
• $1^{st}$-year calc: $\int_a^b f(x)\,dx$ "Rieman $\int$"




$$f(x) \approx f(x_1)\,\chi_{[a,x_1)} + f(x_2)\,\chi_{[x_1,x_2)} + f(x_3)\,\chi_{[x_2,x_3)} + f(x_4)\,\chi_{[x_3,b]}$$

$$\Longrightarrow$$

$$\int_a^b f \approx f(x_1)(x_1-a) + \cdots + f(x_4)(b-x_3)$$




$$f(x) \approx y_1 \chi_{f^{-1}([y_0,y_1))} + y_2 \chi_{f^{-1}([y_1,y_2))} + y_3 \chi_{f^{-1}([y_2,y_3))} + y_4 \chi_{f^{-1}([y_3,y_4])}$$

"Lebesgue integral"

$$\Rightarrow \int_a^b f \approx y_1 \left|f^{-1}([y_0,y_1))\right| + y_2 \left|f^{-1}([y_1,y_2))\right| + y_3 \left|f^{-1}([y_2,y_3))\right| + y_4 \left|f^{-1}([y_3,y_4])\right|$$

"length"

- disadvantage of Lebesgue: need to compute "length" of complicated sets $\longrightarrow$ <u>measure theory!</u>
- so what's wrong with Riemann??

① cannot R-integrate "enough" fns. Eg, a sequence of fns $\{f_n(x)\}_{n=1}^{\infty}$, $f_n$ R-integrable, but

$$f(x) := \lim_{n\to\infty} f_n(x) \text{ is not}$$

② $L$-$\int$ is more general than $R$-$\int$:

a) change the measure on $\mathbb{R}$ $\longrightarrow$ Rieman-Stieltjes $\int$

$\longrightarrow$ summation

$\longrightarrow$ b) works not just on $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^n$, etc..., but on more general sets, $X$ $\longrightarrow$ $\int_X f$

( start of prob. thy:

$X = \{\text{outcomes}\}$

$\int_X f =$ exp. of random var. )