

Math 257/316 Section 202 Midterm 2 March 13

Name: _____ Student #: _____

2 questions; 50 minutes; max = 30 points;

Instructions: no calculators, books, notes, or electronics. Show and explain all your work.

1. (15 points) Find the solution of the non-homogeneous initial-boundary-value problem

$$\begin{cases} u_t = u_{xx} + \sin(x), & 0 < x < \pi, t > 0, \\ u(0, t) = 0, \quad u(\pi, t) = 1 \\ u(x, 0) = 0 \end{cases}$$

• 1st. find a steady-state $V(x)$: $0 = V_{xx} + \sin x \Rightarrow V_x = \cos x + A \Rightarrow V = \sin x + Ax + B$

$$\text{BCs} \left\{ \begin{array}{l} \bullet 0 = V(0) = B \\ \bullet 1 = V(\pi) = A\pi \end{array} \right\} \Rightarrow V(x) = \frac{x}{\pi} + \sin(x)$$

- the problem for $w(x, t) = u(x, t) - V(x)$ is

$$\left\{ \begin{array}{l} w_t = w_{xx}, \quad 0 < x < \pi, t > 0 \\ w(0, t) = 0 = w(\pi, t) \\ w(x, 0) = u(x, 0) - V(x) = -\sin(x) - \frac{x}{\pi} \end{array} \right\} \quad \text{i.e. HE with O BCs}$$

• so: $w(x, t) = \sum_{k=1}^{\infty} b_k \sin(kx) e^{-k^2 t}$ with $b_k = \frac{2}{\pi} \int_0^{\pi} \sin(kx) \left[-\sin(x) - \frac{x}{\pi} \right] dx$

$$\int_0^{\pi} \sin(kx) \sin(kx) dx = \begin{cases} 0 & k \neq 1 \\ \frac{\pi}{2} & k=1 \end{cases} \Rightarrow u(x, t) = \sin(x) + \frac{1}{\pi} \sin(x) e^{-t}$$

$$\begin{aligned} \int_0^{\pi} \sin(kx) dx &= -\frac{\cos(kx)}{k} \Big|_0^{\pi} + \frac{1}{k} \int_0^{\pi} \cos(kx) dx \\ &= -\frac{\pi}{k} \cos(k\pi) + \frac{1}{k^2} \sin(kx) \Big|_0^{\pi} \\ &= -\frac{\pi}{k} (-1)^k \end{aligned}$$

2. (15 points) For the following inhomogeneous heat equation problem with mixed BCs:

$$\begin{cases} u_t = u_{xx} - xe^{-t}, & 0 < x < 1, t > 0 \\ u(0, t) = 1, \quad u_x(1, t) = 1 \\ u(x, 0) = 0 \end{cases}$$

- (a) (7 points) What are the "eigenfunctions" for this problem? Write the form of the solution $u(x, t)$ using an eigenfunction expansion (but do not try to compute any coefficients).
- (b) (8 points) Write a finite-difference numerical scheme to approximate the solution $u(x, t)$, using step sizes Δx (in space) and Δt (in time), with $x_n = n\Delta x$, $t_k = k\Delta t$, $n = 0, 1, 2, \dots, N$; $k = 0, 1, 2, \dots$. Include formulas for computing the approximate values $u_n^k \approx u(x_n, t_k)$, and for incorporating the BCs and IC.

(a) the "eigenfunctions" are solns. of the "X problem" from separation of variables applied to the homogeneous problem (no source term and 0 BCs):

$$\begin{cases} X'' = -\lambda^2 X \\ X(0) = 0 \\ X'(0) = 0 \end{cases} \Rightarrow X(x) = A \cos(\lambda x) + B \sin(\lambda x) \quad \boxed{X_{lk}(x) = \sin(\pi(k+\frac{1}{2})x) = \sin(\pi(k+\frac{1}{2})x)}$$

$$\bullet 0 = X(0) = A \quad \bullet 0 = X'(0) = \lambda B \cos(\lambda) \Rightarrow \lambda = \lambda_k = \frac{\pi}{2} + \pi k \quad k = 0, 1, 2, \dots$$

- to take care of the BCs, use a steady-state (dropping the source term)

$$\begin{cases} V''(x) = 0 \\ V(0) = 1, \quad V'(0) = 1 \end{cases} \Rightarrow V(x) = x+1, \text{ so}$$

$$U(x, t) = x+1 + \sum_{k=0}^{\infty} c_{lk}(t) \sin\left(\pi(k+\frac{1}{2})x\right)$$

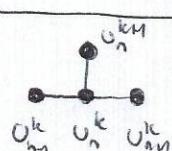
(b) • using the FD approx's $U_t(x, t) \approx \frac{U(x, t+\Delta t) - U(x, t)}{\Delta t}$, $U_{xx}(x, t) \approx \frac{U(x+\Delta x, t) + U(x-\Delta x, t) - 2U(x, t)}{(\Delta x)^2}$
the PDE reads

$$\frac{U_n^{k+1} - U_n^k}{\Delta t} = \frac{U_{n+1}^k + U_{n-1}^k - 2U_n^k}{(\Delta x)^2} - x_n e^{-tk} \Rightarrow U_n^{k+1} = U_n^k + \frac{\Delta t}{(\Delta x)^2} [U_{n+1}^k + U_{n-1}^k - 2U_n^k] - x_n e^{-tk}$$

• for the (IC): $U_n^0 = 0 \quad n = 0, 1, 2, \dots, N$

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• left (BC): $U_0^k = 1, \quad k = 1, 2, 3, \dots$



• scheme for computing U_n^{k+1} , given $U_n^k, U_{n+1}^k, U_{n-1}^k$

• for the right (BC), use $U_{N+1}^k \approx \frac{U(1+\Delta x, t) - U(1-\Delta x, t)}{2\Delta x} = \frac{U_{N+1}^k - U_{N-1}^k}{2\Delta x}$ extra "ghost" point to the right of the physical grid

$$\Rightarrow U_{N+1}^k = U_{N-1}^k + 2\Delta x, \quad k = 1, 2, 3, \dots$$

(with U_N^k computed using the scheme above)