

Instructions. The exam lasts 50 minutes. Calculators are not allowed. A formula sheet is attached.

**[38 marks]**

1. Consider the following problem for the heat equation with a source term, and non-homogeneous Neumann boundary conditions:

$$u_t = u_{xx} + 1 \quad 0 < x < 1, \quad t > 0$$

$$u_x(0, t) = 0, \quad u_x(1, t) = 2$$

$$u(x, 0) = x^2 + 1 - 2 \cos(\pi x).$$

**8** (a) Find a particular solution of the PDE and boundary conditions of the form  $v(x, t) = ax^2 + bx + ct$ .

**12** (b) Find the solution  $u(x, t)$  of the full problem.

**8** (c) Write a finite difference approximation to the PDE (i.e.  $u_t = u_{xx} + 1$ ), and also write finite difference approximations for the boundary conditions. In case you need it, the Taylor expansion formula is  $f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2}f''(x)(\Delta x)^2 + O((\Delta x)^3)$ .

$$(a) v = ax^2 + bx + ct; \quad v_t = v_{xx} + 1 \Rightarrow c = 2a + 1$$

$$v_{xL} = 2ax + b \quad \circ \quad 0 = v_{xL}(0, t) = b$$

$$\circ \quad 2 = v_{xL}(1, t) = 2a + b = 2a \Rightarrow a = 1 \Rightarrow c = 3$$

$$\text{so } \boxed{v(x, t) = x^2 + 3t}$$

(b) write  $u(x, t) = v(x, t) + w(x, t)$  so that

$$\therefore \text{so } w(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) e^{-n\pi^2 t}$$

and the IC requires

$$1 - 2\cos(\pi x) = w(x, 0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x).$$

$$\text{Thus } a_0 = 2, \quad a_1 = -2, \quad \text{and} \quad a_2 = a_3 = a_4 = \dots = 0.$$

$$\text{So } \boxed{u(x, t) = x^2 + 3t + 1 - 2\cos(\pi x) e^{-\pi^2 t}}$$

$$\left. \begin{aligned} w_t &= w_{xx}, \quad 0 < x < 1, \quad t > 0 \\ w_x(0, t) &= 0 = w_x(1, t) \\ w(x, 0) &= u(x, 0) - v(x, 0) \\ &= x^2 + 1 - 2\cos(\pi x) - x^2 \\ &= 1 - 2\cos(\pi x) \end{aligned} \right\}$$

$$(c) \text{ approximate } \frac{\partial u}{\partial t}(x, t) \approx \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} \approx \frac{u(x + \Delta x, t) + u(x - \Delta x, t) - 2u(x, t)}{(\Delta x)^2}$$

so a f.d. approx. of the PDE is.

$$\boxed{\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \frac{u(x + \Delta x, t) + u(x - \Delta x, t) - 2u(x, t)}{(\Delta x)^2} + }$$

For the BCs :

$$0 = \frac{\partial u}{\partial x}(0, t) \approx \frac{u(+\Delta x, t) - u(-\Delta x, t)}{2\Delta x} \Rightarrow \boxed{u(-\Delta x, t) = u(\Delta x, t)}$$

$$2 = \frac{\partial u}{\partial x}(1, t) \approx \frac{u(1 + \Delta x, t) - u(1 - \Delta x, t)}{2\Delta x} \stackrel{1}{\Rightarrow} \boxed{u(1 + \Delta x, t) = u(1 - \Delta x, t) + 4\Delta x}$$

[20 marks]

2. Use the method of separation of variables to solve the following initial boundary value problem for the wave equation with zero Neumann BCs:

$$u_{tt} = u_{xx} \quad 0 < x < \pi, \quad t > 0$$

$$u_x(0, t) = 0 = u_x(\pi, t)$$

$$u(x, 0) = 0, \quad u_t(x, 0) = \begin{cases} 1 & 0 \leq x < \pi/2 \\ 0 & \pi/2 < x \leq \pi \end{cases} \therefore g(x)$$

$$\circ u(x, t) = X(x)T(t) \Rightarrow \frac{T''}{T} = \frac{X''}{X} = -\lambda$$

$$\circ X \text{ problem: } \begin{cases} X''(x) = -\lambda X(x) \\ X'(0) = 0 = X'(\pi) \end{cases} \Rightarrow \lambda = \lambda_n = n^2, \quad n = 0, 1, 2, \dots$$

$$X = X_n(x) = \cos(nx)$$

$$\circ T \text{ problem: } T''(t) = -\lambda_n T(t) \therefore n=0 \Rightarrow T''=0 \Rightarrow T(t) = \frac{A_0}{2} + \frac{B_0}{2} t$$

$$\therefore n \geq 1 \Rightarrow T'' = -n^2 T \therefore T(t) = A_n \cos(nt) + B_n \sin(nt)$$

$$\circ \text{ general solution: } u(x, t) = \frac{A_0}{2} + \frac{B_0}{2} t + \sum_{n=1}^{\infty} [A_n \cos(nt) + B_n \sin(nt)] \cos(nx)$$

$\circ$  initial conditions:

$$\circ 0 = u(x, 0) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) \Rightarrow A_0 = A_1 = A_2 = \dots = 0$$

$$\circ g(x) = u_t(x, 0) = \frac{B_0}{2} + \sum_{n=1}^{\infty} n B_n \cos(nx)$$

$$\Rightarrow B_0 = \frac{2}{\pi} \int_0^{\pi} g(x) dx = \frac{2}{\pi} \cdot \frac{\pi}{2} = \cancel{\cancel{1}}$$

$$\text{and for } n \geq 1, \quad n B_n = \frac{2}{\pi} \int_0^{\pi} g(x) \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} \cos(nx) dx = \frac{2}{\pi n} \sin(n\pi/2)$$

$$\Rightarrow B_n = \frac{2}{\pi n^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \boxed{u(x, t) = \frac{1}{2} t + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) \sin(nt) \cos(nx)}$$