

Math 257/316 Section 202 Midterm 1 February 6

2 questions; 50 minutes; max = 30 points

1. Consider this second order, linear, homogeneous ODE:

$$2(x-1)y'' + y' + y = 0.$$

- (a) Find the general solution in the form of a power series based at $x_0 = 0$ (just find the first four non-zero terms). [8 points]
- (b) Find (only!) the first two terms of a series solution based at $x_0 = 1$ (and valid for $x > 1$) satisfying $y(1) = 1$, $\lim_{x \rightarrow 1^+} \sqrt{x-1}y'(x) = 1$. [8 points]

(a) $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$0 = 2xy'' - 2y' + y = \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} - \underbrace{\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2}}_{\text{can replace by } n=1 \text{ since } n \geq 1 \text{ term } \approx 0} + \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$

(or else collect x^0 terms separately)

$$= \sum_{n=1}^{\infty} (2n(n-1)a_n - 2(n+1)n a_{n+1} + n a_n + a_{n-1}) x^{n-1} = \sum_{n=1}^{\infty} (-2(n+1)n a_{n+1} + (2n^2-n)a_n + a_{n-1}) x^{n-1}$$

$\Rightarrow \boxed{a_{n+1} = \frac{n(2n-1)a_n + a_{n-1}}{2n(n+1)} \quad n=1, 2, 3, \dots} \Rightarrow a_0, a_1 \text{ free; } a_2 = \frac{a_1 + a_0}{4};$

$a_3 = \frac{6a_2 + a_1}{12} = \frac{3/2(a_1 + a_0) + a_1}{12} = \frac{5a_1 + 3a_0}{24}$

$\Rightarrow \boxed{y(x) = a_0 + a_1 x + \frac{1}{4}(a_0 + a_1)x^2 + \frac{1}{24}(3a_0 + 5a_1)x^3 + \dots}$

(b) $y'' + \frac{1}{2(x-1)}y' + \frac{1}{2(x-1)}y = 0$ so $x_0 = 1$ is a R.S.P. since $\begin{cases} (x-1) \cdot \frac{1}{2(x-1)} = \frac{1}{2} =: \alpha \\ (x-1)^2 \cdot \frac{1}{2(x-1)} = \frac{1}{2}(x-1) \xrightarrow{x \rightarrow 1} 0 =: \beta \end{cases}$

and the indicial equation is $0 = r(r-1) + \frac{1}{2}r = r(r-\frac{1}{2})$

with roots $r_1 = \frac{1}{2}$, $r_2 = 0$. So there is a pair of fund. solns. (for $x > 1$):

$$\left. \begin{array}{l} y_1(x) = (x-1)^{\frac{1}{2}} + a_1(x-1)^{\frac{3}{2}} + \dots \\ y_2(x) = 1 + b_1(x-1) + \dots \end{array} \right\} \text{ and the general soln. is } y(x) = C_1 y_1(x) + C_2 y_2(x) = \left(\begin{array}{l} + C_1(x-1)^{\frac{1}{2}} + b_1(x-1) + C_2 \\ + C_1(x-1)^{\frac{3}{2}} + C_2(x-1)^{\frac{5}{2}} + \dots \end{array} \right)$$

$\bullet 1 = y_1(1) = C_2$

$\bullet 1 = \lim_{x \rightarrow 1^+} (x-1)^{\frac{1}{2}} y_1(x) = \lim_{x \rightarrow 1^+} (x-1)^{\frac{1}{2}} \left[\frac{1}{2} C_1(x-1)^{-\frac{1}{2}} + C_2 b_1(x-1) + \dots \right] = \frac{1}{2} C_1 \Rightarrow C_1 = 2$

so $\boxed{y(x) = 1 + 2(x-1)^{\frac{1}{2}} + \dots}$

2. Consider the initial-boundary-value problem

$$\begin{cases} u_{tt} + 2u_t = u_{xx}, & 0 < x < 1, \quad t > 0, \\ u(0, t) = 0, \quad u(1, t) = 0 \\ u(x, 0) = f(x), \quad u_t(x, 0) = g(x) \end{cases}$$

- (a) Write clearly the "X problem" and the "T problem" that arise when finding separated variables solutions $u(x, t) = X(x)T(t)$ of the PDE and BC. [5 points]
- (b) The solutions of the "X problem" are $X_k(x) = \sin(k\pi x)$, $k = 1, 2, 3, \dots$. Solve the "T problem" and write the general solution of the PDE and BC [5 points]
- (c) Find the solution when $f(x) = 0$, $g(x) = \sin(4\pi x)$. [4 points]

$$(a) \quad 0 = XT \Rightarrow XT'' + 2XT' = X''T \Rightarrow \frac{T''}{T} + \frac{2T'}{T} = \frac{X''}{X} = \mu = \text{constant}$$

\Rightarrow

$$\boxed{\begin{array}{l} X \text{ problem} \\ \left\{ \begin{array}{l} X'' = \mu X \\ X(0) = 0 = X(1) \end{array} \right. \end{array}}$$

$$\boxed{\begin{array}{l} T \text{ problem} \\ T'' + 2T' - \mu T = 0 \end{array}}$$

$$(b) \quad \mu_k = \frac{X''_k}{X_k} = \frac{-k^2\pi^2 \sin(k\pi x)}{\sin(k\pi x)} = -k^2\pi^2, \quad \text{so the ODE for } T_k \text{ is} \quad T'' + 2T' + k^2\pi^2 T = 0$$

- this is constant coeff. with characteristic eqn. $0 = r^2 + 2r + k^2\pi^2$
- $\Rightarrow r = -1 \pm \sqrt{1-k^2\pi^2} = -1 \pm \lambda \sqrt{k^2\pi^2 - 1}$, so $T_k(t) = e^{-t} [A_k \cos(\sqrt{k^2\pi^2 - 1}t) + B_k \sin(\sqrt{k^2\pi^2 - 1}t)]$

The general soln. of PDE + BC is

$$\boxed{u(x, t) = \sum_{k=1}^{\infty} e^{-t} [A_k \cos(\sqrt{k^2\pi^2 - 1}t) + B_k \sin(\sqrt{k^2\pi^2 - 1}t)] \sin(k\pi x)}$$

$$(c) \quad \bullet \quad 0 = f(x) = u(x, 0) = \sum_{k=1}^{\infty} A_k \sin(k\pi x) \Rightarrow A_k = 0 \text{ for all } k$$

$$\bullet \quad u_t = \sum_{k=1}^{\infty} B_k e^{-t} [\sqrt{k^2\pi^2 - 1} \cos(\sqrt{k^2\pi^2 - 1}t) - \sin(\sqrt{k^2\pi^2 - 1}t)] \sin(k\pi x)$$

$$\bullet \quad \sin(4\pi x) = g(x) = u_t(x, 0) = \sum_{k=1}^{\infty} B_k \sqrt{k^2\pi^2 - 1} \sin(k\pi x) \Rightarrow \begin{cases} B_4 = \frac{1}{\sqrt{16\pi^2 - 1}} \\ B_k = 0 \text{ for } k \neq 4 \end{cases}$$

$$\Rightarrow \boxed{u(x, t) = \frac{1}{\sqrt{16\pi^2 - 1}} e^{-t} \sin(\sqrt{16\pi^2 - 1}t) \sin(4\pi x)}$$