Math 257/316 Section 201 Midterm 1 Solutions

Total = 50 points [There are 2 questions.]

Problem 1. Consider the second order differential equation:

$$2x^2y'' + (3x - x^2)y' - y = 0.$$

Find the first three terms of a (non-zero) solution, in the form of a series based at $x_0 = 0$ (and valid for $x > 0$), satisfying $\lim_{x\to 0+} y(x) = 0$. [20 points]

Dividing through by $2x^2$, the coefficients are $p(x) = (3x - x^2)/(2x^2) = 3/(2x) - 1/2$ and $q(x) = -1/(2x^2)$ which are both singular at $x = 0$. Since $xp(x) = 3/2 - x/2 \to 3/2$ as $x \to 0$ and $x^2q(x) = -1/2 \to 0$ as $x \to 0$, we see that 0 is a regular singular point, and we should look for a Frobenius series solutions. At this point, it would save time to find the roots of the indicial equation $r(r-1) + (3/2)r - 1/2 = 0$ and choose the one that gives the right behaviour (i.e. $r = 1/2$). But suppose we didn't think of that. Write

$$y = \sum_{k=0}^{\infty} a_k x^{r+k}, \quad y' = \sum_{k=0}^{\infty} (r+k)a_k x^{r+k-1}, \quad y'' = \sum_{k=0}^{\infty} (r+k)(r+k-1)a_k x^{r+k-2},$$

and substitute into the ODE to get

$$\begin{aligned} 0 &= 2x^2y'' + 3xy' - x^2y' - y \\ &= \sum_{k=0}^{\infty} 2(r+k)(r+k-1)a_k x^{r+k} + \sum_{k=0}^{\infty} 3(r+k)a_k x^{r+k} - \sum_{k=0}^{\infty} (r+k)a_k x^{r+k+1} - \sum_{k=0}^{\infty} a_k x^{r+k} \end{aligned}$$

Shift the index in the third series,

$$\sum_{k=0}^{\infty} (r+k)a_k x^{r+k+1} = \sum_{k=1}^{\infty} (r+k-1)a_{k-1} x^{r+k},$$

separate out all the $k = 0$ terms, and collect the rest together to find

$$0 = [2r(r-1) + 3r - 1]a_0 x^r + \sum_{k=1}^{\infty} \{[(r+k)(2(r+k-1)+3) - 1]a_k - (r+k-1)a_{k-1}\}$$

The first term gives the indicial equation

$$0 = 2r^2 + r - 1 = (2r-1)(r+1)$$

whose roots are $r = 1/2$ *and* $r = -1$. *Since we seek a solution with* $y(x) \to 0$ *as* $x \to 0+$, *we exclude* $r = -1$ *(which corresponds to leading term* x^{-1} *which blows up at* 0*) and set* $r = 1/2$ *(which corresponds to leading term* $x^{1/2}$, *which indeed tends to* 0 *as* $x \to 0+$*). The remaining terms provide the recurrence relation (in which we now take* $r = 1/2$*):*

$$[(k+1/2)(2(k-1/2)+3)-1]a_k - (k-1/2)a_{k-1} = 0 \qquad \Longrightarrow \; a_k = \frac{(k-1/2)a_{k-1}}{(2k+2)(k+1/2)-1}$$

for $k = 1, 2, 3, 4, \ldots$. *And* a_0 *is free. Since we seek only one such (non-zero) solution, let's set* $a_0 = 1$. *Then*

$$a_1 = \frac{1/2}{5} = \frac{1}{10}, \quad a_2 = \frac{3/2}{14}a_1 = \frac{3}{280},$$

and so

$$y(x) = x^{1/2} + \frac{1}{10}x^{3/2} + \frac{3}{280}x^{5/2} + \cdots .$$

Problem 2.

a) Use the method of separation of variables to find the most general solution of the partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \; t > 0,$$

with the "insulating" (or "no-flux") boundary conditions: $\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(1,t) = 0$ (show all your work, and consider the cases of positive, zero, and negative separation constant). [25 points]

Separating variables $u(x,t) = X(x)T(t)$ *leads to*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \Longrightarrow \qquad XT' = X''T$$

and dividing through by XT,

$$\frac{T'}{T} = \frac{X''}{X} = \; constant \; =: \lambda$$

(each side must be constant since they depend on different variables). The "X problem"

$$X'' = \lambda X, \quad X'(0) = 0, \quad X'(1) = 0$$

inherits its boundary conditions from those of the original PDE problem (note it is X' *not* X *which should vanish at* $x = 0$ *and* $x = 1$*). Cases:*

1. *$\lambda > 0$: the general solution of the ODE is $X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$, so $X'(x) = \sqrt{\lambda}(c_1 e^{\sqrt{\lambda}x} - c_2 e^{-\sqrt{\lambda}x})$. $0 = X'(0) = \sqrt{\lambda}(c_1 - c_2)$ implies $c_1 = c_2$. Then $0 = X'(1) = \sqrt{\lambda}c_1(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}})$, which can only be satisfied if $c_1 = 0$, and so $c_2 = 0$. Thus no non-zero solutions in this case.*
2. *$\lambda = 0$: then $X = c_1 x + c_2$, $X' = c_1$. So we need $0 = X'(0) = c_1 = X'(1)$. Thus $X(x) = c_2 =$ constant is a non-zero solution with $\lambda = 0$.*
3. *$\lambda < 0$: the general solution is $X(x) = c_1 \sin(\sqrt{-\lambda}x) + c_2 \cos(\sqrt{-\lambda}x)$, so $X'(x) = \sqrt{-\lambda}(c_1 \cos(\sqrt{-\lambda}x) - c_2 \sin(\sqrt{-\lambda}x))$. The BCs give $0 = X'(0) = \sqrt{-\lambda}c_1$, so $c_1 = 0$, and then $0 = X'(1) = -c_2\sqrt{-\lambda}\sin(\sqrt{-\lambda})$. So for a non-zero solution, we require*

$$\sin(\sqrt{-\lambda}) = 0 \quad \Longrightarrow \quad \sqrt{-\lambda} = k\pi, \quad k = 1, 2, 3, \ldots.$$

So we have our solutions of the X problem:

$$X_0(x) = 1 \text{ with } \lambda = \lambda_0 := 0, \quad X_k(x) = \cos(k\pi x) \text{ with } \lambda = \lambda_k := -k^2\pi^2, \quad k = 1, 2, 3, \ldots.$$

(and any constant multiples of these). The solutions of the corresponding "T problem"

$$T' = \lambda_k T = -k^2\pi^2 T$$

are constant multiples of $e^{-k^2\pi^2 t}$. Combining with the corresponding X, we have product solutions (of both the heat equation and the given boundary conditions) of the form

$$u_0(x) = 1, \quad u_k(x) = \cos(k\pi x)e^{-k^2\pi^2 t}, \quad k = 1, 2, 3, \ldots.$$

Finally, the most general solution we can write is an infinite linear combination of these (which still satisfies the boundary conditions – since they are homogeneous – as well as the PDE, since it is linear and homogeneous):

$$u(x,t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\pi x)e^{-k^2\pi^2 t}.$$

b) Explain the behaviour of this solution as $t \to \infty$, and give a physical interpretation in terms of the temperature of a wire, and in terms of a diffusion (continuous limit of a random walk). [5 points]

We see $\lim_{t\to\infty} u(x,t) = a_0$ (constant).

For the temperature of a wire: transfer of heat smooths out temperature differences, and the temperature tends toward a uniform distribution. Since no heat can leave

or enter (due to the insulating boundary conditions), the limiting temperature is in general non-zero (in fact equal to the average of the initial temperature distribution).

For diffusion: the random motion of the particles leads toward a uniform particle distribution. Since particles cannot enter or leave, the limiting distribution is not zero; in fact it is such that the total number of particles (the integral of the density) is the same as at the initial time.