

Instructions. The exam lasts 50 minutes. Calculators are not allowed. A formula sheet is attached.

1. Consider the ODE

$$2x^2y'' + (3x + x^2)y' - y = 0.$$

(a) Verify that the point $x = 0$ is a regular singular point for this ODE. [5 marks]

(b) For $x > 0$, find the first three non-zero terms of a series solution about $x = 0$ for which $\lim_{x \rightarrow 0+} y(x) = 0$. [20 marks]

(a) $y'' + \underbrace{\left[\frac{3}{2x} + \frac{1}{2}\right]}_{p(x)} y' + \underbrace{\left[-\frac{1}{2x^2}\right]}_{q(x)} y = 0$

- both p and q are singular at $x=0$
- $\Rightarrow 0$ is a singular point
- $\lim_{x \rightarrow 0} xp(x) = \frac{3}{2}$ & $\lim_{x \rightarrow 0} xq(x) = -\frac{1}{2}$ are finite $\Rightarrow 0$ is a regular singular point

(b) Take $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$\Rightarrow y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}.$$

Then the ODE is

$$0 = 2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + 3x \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} + x^2 \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$= \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)a_n x^{n+r} + \underbrace{\sum_{n=0}^{\infty} (n+r)a_n x^{n+r}}_{(n+r-1)\sum_{n=1}^{\infty} (n+r-1)a_{n-1} x^{n+r}}$$

$$= \underbrace{[2r(r-1) + 3r - 1]a_0 x^r}_{2r^2+r-1} + \sum_{n=1}^{\infty} \underbrace{\left\{ [(n+r)[2(n+r-1)+3]-1]a_n + (n+r-1)a_{n-1} \right\} x^{n+r}}_{[(n+r)[2(n+r-1)+1]-1]a_n + (n+r-1)a_{n-1}}$$

so we need $0 = 2r^2+r-1 = (2r-1)(r+1)$ (indicial equation) $\Rightarrow r=-1$ or $r=\frac{1}{2}$

since $\lim_{x \rightarrow 0+} y(x) = 0$, we want the solution behaving like $x^{\frac{1}{2}}$ (not x^{-1})
 \Rightarrow we choose $r = \frac{1}{2}$

then the recurrence relation is $\{(n+\frac{1}{2})[2n+2]-1\}a_n + (n-\frac{1}{2})a_{n-1} = 0$

$$\Rightarrow a_n = \frac{-(n-\frac{1}{2})a_{n-1}}{2n^2+3n}. \quad \text{Take } a_0 = 1. \quad \text{Then } a_1 = \frac{-\frac{1}{2}}{5} = -\frac{1}{10}.$$

$$\Rightarrow \boxed{y(x) = x^{\frac{1}{2}} - \frac{1}{10}x^{\frac{3}{2}} + \frac{3}{280}x^{\frac{5}{2}} + \dots}$$

and $a_2 = \frac{-3/2}{14}a_1 = \frac{3}{280}$

2. The temperature $u(x, t)$ along a wire $0 \leq x \leq \pi/2$ with the left end insulated and the right end held at zero temperature satisfies:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, & 0 < x < \pi/2, \quad t > 0 \\ \frac{\partial u}{\partial x}(0, t) = 0, \quad u(\pi/2, t) = 0 \end{cases}$$

- (a) Use separation of variables to find the most general solution of the PDE and boundary conditions (consider all cases for the separation constant: positive, zero, and negative). [20 marks]
- (b) Find the solution satisfying the initial condition $u(x, 0) = \cos(x) - \cos(5x)$. [5 marks]

$$(a) u(x, t) = X(x)T(t) \Rightarrow XT' = X''T \Rightarrow \frac{T'}{T} = \frac{X''}{X} = -\lambda, \text{ a constant}$$

X problem: $\begin{cases} X'' = -\lambda X \\ X'(0) = 0, \quad X(\pi/2) = 0 \end{cases}$

from the BCs $\begin{cases} \text{if } \lambda < 0, \quad X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x} \\ \text{and } X' = \sqrt{-\lambda}(Ae^{\sqrt{-\lambda}x} - Be^{-\sqrt{-\lambda}x}) \\ \text{So } 0 = X'(0) = \sqrt{-\lambda}(A - B) \Rightarrow A = B \\ \Rightarrow X(x) = 2A \cosh(\sqrt{-\lambda}x) \\ \text{and } 0 = X(\pi/2) = 2A \cosh(\sqrt{-\lambda}\pi/2) \\ \Rightarrow A = 0 \Rightarrow X \equiv 0 \Rightarrow \text{no non-zero solution} \end{cases}$

$\begin{cases} \text{if } \lambda = 0, \quad X(x) = Ax + B, \quad X'(0) = A \\ \text{so } 0 = X'(0) = A \text{ and then} \\ 0 = X(\pi/2) = B \Rightarrow X \equiv 0 \\ \Rightarrow \text{no non-zero solutions} \end{cases}$

$\begin{cases} \text{so } \lambda > 0: \Rightarrow X(x) = A \sin(\sqrt{\lambda}x) + B \cos(\sqrt{\lambda}x), \quad X'(x) = \sqrt{\lambda}(A \cos(\sqrt{\lambda}x) - B \sin(\sqrt{\lambda}x)) \\ 0 = X'(0) = \sqrt{\lambda}A \Rightarrow A = 0 \Rightarrow X(x) = B \cos(\sqrt{\lambda}x) \\ 0 = X(\pi/2) = B \cos(\sqrt{\lambda}\pi/2) \Rightarrow \sqrt{\lambda}\pi/2 = \pi/2 + n\pi \quad n=0,1,2,3,\dots \\ \Rightarrow \sqrt{\lambda} = 1 + 2n \Rightarrow \lambda = (1+2n)^2, \quad n=0,1,2,3,\dots \end{cases}$

T problem: take $\lambda = (1+2n)^2 \Rightarrow T'(t) = -(1+2n)^2 T(t) \Rightarrow T(t) = (\text{const}) e^{-(1+2n)^2 t}$

so our product solutions are $(\text{const.}) \cos((1+2n)x) e^{-(1+2n)^2 t}, \quad n=0,1,2,3,\dots$
and the general solution is $\boxed{u(x, t) = \sum_{n=0}^{\infty} c_n \cos((1+2n)x) e^{-(1+2n)^2 t}}$

(b) We require $\cos(x) - \cos(5x) = u(x, 0) = \sum_{n=0}^{\infty} c_n \cos((1+2n)x)$
and inspection gives $\begin{cases} c_0 = 1, \quad c_2 = -1, \\ c_1 = c_3 = c_4 = c_5 = \dots = 0 \end{cases}$

$\Rightarrow \boxed{u(x, t) = \cos(x)e^{-t} - \cos(5x)e^{-25t}}$