$\boxed{\text{Nov26 } 2012}$

$$x^2 X''(x) + 2x X'(x) + \lambda X(x) = 0. \qquad X(1)=0, \; X(2)=0.$$

How do we determine the eigenvalues?

Consider: $x^2 \phi'' + 2x\phi' + \lambda\phi = 0.$

$\phi(x) = x^m$

$\phi' = m x^{m-1}$

$\phi'' = m(m-1)x^{m-2}$

$\Rightarrow x^2 m(m-1)x^{m-2} + 2x\, m x^{m-1} + \lambda x^m = 0.$

$\Rightarrow m^2 - m + 2m + \lambda = 0$

$\Rightarrow m^2 + m + \lambda = 0. \Rightarrow m = \dfrac{-1 \pm \sqrt{1-4\lambda}}{2}$ ⊛

$\boxed{\phi(1)=0, \phi(2)=0}$

$= m_1, m_2 \leftarrow$ Two roots

$$\phi(x) = C_1 x^{m_1} + C_2 x^{m_2}$$

$\phi(1)=0 \Rightarrow C_1 + C_2 = 0 \Rightarrow C_2 = -C_1.$

$\phi(2)=0 \Rightarrow C_1 2^{m_1} - C_1 2^{m_2} = 0$

$\Rightarrow C_1(2^{m_1} - 2^{m_2}) = 0$

$\Rightarrow 2^{m_1} - 2^{m_2} = 0 \Rightarrow 2^{m_1} = 2^{m_2}$

$\Rightarrow 2^{m_1 - m_2} = 1 \leftarrow 2^{m_2} \cdot 2^{-m_2} = 1$

$\Rightarrow e^{(m_1-m_2)\ln 2} = 1 = e^{2\pi i n}$

$\boxed{\text{NOTE: } 2^{m_1-m_2} = e^{\ln(2^{m_1-m_2})}}$

$e^{2i\pi} = 1$, so $e^{2\pi i n} = 1$

$\Rightarrow (m_1 - m_2)\ln 2 = 2\pi i n$

$m_1 - m_2 = \dfrac{2\pi i n}{\ln 2}$

But $m_1 - m_2 = \sqrt{1-4\lambda}$, using ⊛ above.

$m_1 - m_2 = \sqrt{-1(4\lambda - 1)} \leftarrow m_1 - m_2 \neq 0 \Leftrightarrow 4\lambda - 1 \neq 0.$

$\boxed{\lambda \neq \frac{1}{4}}$ $\Rightarrow i\sqrt{4\lambda-1} = \dfrac{2\pi i n}{\ln 2} \Rightarrow \sqrt{4\lambda - 1} = \dfrac{2\pi n}{\ln 2}$

$\Rightarrow 4\lambda - 1 = \dfrac{4\pi^2 n^2}{(\ln 2)^2} \Rightarrow \boxed{\lambda_n = \dfrac{1}{4} + \dfrac{\pi^2 n^2}{(\ln 2)^2}}$

NOTE: if $m_1 = m_2$, there are no solutions to $\phi(x) = x^m$ that satisfy $\phi(1) = \phi(2) = 0$.