

Lecture 13 (Oct. 9-10)

14.7: Maximum and Minimum Values

Sample problem to keep in mind: find the dimensions to minimize the cost of an aquarium (rectangular cube, open top) with a slate base and glass sides, if the cost of slate is 5 times the cost of glass.

Recall: max/min value situation in one dimension:

Definition: Let $f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$.

1. f has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ for all (x, y) in some (possibly small) disk centred at (a, b) .
2. f has a **local minimum** at (a, b) if $f(x, y) \geq f(a, b)$ for all (x, y) in some (possibly small) disk centred at (a, b) .
3. f has an **absolute maximum** (resp. **absolute minimum**) at (a, b) if $f(x, y) \leq f(a, b)$ (resp. $f(x, y) \geq f(a, b)$) for all (x, y) in D .

Theorem: Suppose f has a local maximum or minimum at (a, b) , and suppose $\nabla f(a, b)$ exists. Then $\nabla f(a, b) = \mathbf{0}$.

Proof:

Remark: $\nabla f(a, b) = \mathbf{0}$ means the tangent plane is horizontal.

Definition: We say (a, b) is a **critical point** of f if $\nabla f(a, b) = \mathbf{0}$, or if one of the partials fails to exist at (a, b) .

So we see that if (a, b) is a local max./min. of f , then (a, b) must be a critical point. The reverse implication is false in general.

Example: $f(x, y) = x^2 + 3y^2$

Example: $f(x, y) = xy$

Theorem: (“second derivatives test”). Suppose $\nabla f(a, b) = \langle 0, 0 \rangle$ (so (a, b) is a critical point of f), and suppose that the second partials of f are continuous on some disk centred at (a, b) . Set

$$D := f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

Then

- if $D > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b)
- if $D > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b)
- if $D < 0$, f has neither a local max. nor a local min. at (a, b) .

Remark:

1. if $D < 0$, (a, b) is called a **saddle point** of f
2. if $D = 0$, the test gives no information
3. note that

$$D = \det \begin{pmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{pmatrix}.$$

Example: Find and classify the critical points of $f(x, y) = x^2 + y^2 + x^2y + 4$.

Example: Find the dimensions of an aquarium (rectangular cube, open top) of given volume which minimizes the cost of the slate (for the base) and the glass (for the sides), supposing the slate costs 5 times as much as the glass.

Absolute maximum and minimum values

Definition:

- a set D is called **bounded** if it can be contained in a disk (2D) or ball (3D)
- a set D is called **closed** if it contains all of its boundary points.

Theorem: [*Extreme value theorem*] if a function f is continuous on a closed, bounded set D , then it attains an absolute minimum and absolute maximum value on D .

Example: Let us illustrate the necessity of the hypotheses:

Summary: how to find the absolute max/min of a continuous function f on a closed, bounded set D :

1. find the values of f at all critical points
2. find the extreme (max/min) values on the boundary
3. take the overall largest/smallest value from the above

Example: find the absolute max. and min. of $f(x, y) = x^2 + 2xy + 3y^2$ on the set bounded by the triangle with vertices $(-1, 1)$, $(2, 1)$, and $(-1, -2)$.